이학박사 학위논문

Tangential limits of harmonic functions for subordinate Brownian motions

(종속 브라운 운동에 대한 조화함수의 접선극한)

2015년 2월

서울대학교 대학원

수리과학부

강 재 훈

Tangential limits of harmonic functions for subordinate Brownian motions

(종속 브라운 운동에 대한 조화함수의 접선극한)

지도교수 김 판 기

이 논문을 이학박사 학위논문으로 제출함

2014년 10월

서울대학교 대학원

수리과학부

강 재 훈

강 재 훈의 이학박사 학위논문을 인준함

2014년 12월

위 원 장 ________________________ (인)

부 위 원 장 ________________________ (인)

위 원 ________________________ (인)

위 원 ________________________ (인)

위 원 ________________________ (인)

Tangential limits of harmonic functions for subordinate Brownian motions

A dissertation
submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
to the faculty of the Graduate School of
Seoul National University

by

Jaehoon Kang

Dissertation Director : Professor Panki Kim

Department of Mathematical Sciences
Seoul National University

February 2015

Abstract

In this thesis, we study the integral kernel and boundary behavior of harmonic functions for certain non-local operators. First, using elementary calculus only, we give a simple proof that Green function estimates imply the sharp two-sided Poisson kernel estimates for pure-jump subordinate Brownian motions including geometric stable processes. The infinitesimal generators of pure-jump subordinate Brownian motions are non-local operators. Second, we show the existence of tangential limits of regular harmonic functions with respect to such non-local operators in $C^{1,1}$ open sets when the exterior functions are local L^p-Hölder continuous functions of order β.

Key words: subordinate Brownian motion, Green function, Poisson kernel, non-local operator, harmonic function, (non-)tangential limits, Fatou theorem
Student Number: 2011-30090

Contents

Chapter 1

Introduction

Probability has been extensively studied during past several decades. There are a lot of astonishing results, and they are applied in various fields such as biology, economics, physics, and so on. Recently, the Lévy processes are widely studied owing to their importance both in theories and applications.

The researches on Lévy processes have been developed in various subjects. There are many studies on estimates of kernels such as (Dirichlet) heat kernel, Green function, Poisson kernel, and Martin kernel ([7, 8, 10, 22, 23, 25]). There are also a lot of studies on properties of harmonic function with respect to integro-differential operator $\mathcal{A}$, which is the infinitesimal generator of Lévy process, such as (boundary) Harnack inequality, boundary limit of harmonic functions, and regularity of harmonic functions ([16, 18, 21, 24, 27]).

Despite of their importance, the general Lévy processes and corresponding infinitesimal generators are not easy to deal with. In this thesis, we consider the special type of Lévy process, which is called subordinate Brownian motion. A subordinate Brownian motion is a Lévy process which can be obtained by replacing the time of Brownian motion by an independent subordinator. The subordinate Brownian motion covers a large class of Lévy processes.

The purposes of this thesis are to estimate the Poisson kernels for a large class of subordinate Brownian motions and investigate possible tangential approaching regions for harmonic functions with respect to such subordinate Brownian motions in $C^{1,1}$ open sets.

The Poisson kernel is the fundamental solution of boundary value problem. The Poisson kernel has explicit form in some cases. However, in general, it is not possible to derive an explicit Poisson kernel formula for general operators and general open sets. Thus, it is important to obtain the Poisson kernel estimates. Fortunately, by using the recent result on the sharp two-sided estimates on Green function and behavior of jumping kernel in [20], we derive sharp two-sided estimates on the Poisson kernel for subordinate Brownian motion in bounded $C^{1,1}$ open sets under mild assumptions on the Laplace exponent of subordinator ϕ. Indeed, we obtain the same result in bounded open set D which satisfies the cone condition (see Definition 3.1.1) as long as the estimates of Green function and behavior of jumping kernel holds for such D.

We are also interested in the boundary limits of harmonic functions. A study on this topic has a long history. Fatou [14] showed in 1906 that if $f \in L^p(\mathbb{R}^{d-1})$ for $p \in [1, \infty]$, then the Poisson extension u_f of f on the upper half-space has a nontangential limit a.e. on $\mathbb{R}^{d-1}$, which is called Fatou theorem. It is also proved in [28] that the nontangential approach is sharp.

After [14], much have been written in the boundary limits of harmonic functions ([11, 1, 2, 3, 4, 12, 30, 31, 32, 36]). In particular, in [3, 4], the Fatou-type theorem for harmonic functions with respect to α-stable process (see Definition 4.1.1) was discussed. Note that Bass and You [3] showed that the precise analogue of the Fatou theorem for harmonic functions with respect to α-stable process is not true. Thus, in this case, it is necessary to state certain assumptions related to exterior functions to prove the existence of limits at the boundary. Under certain L^p-Hölder continuity assumptions, it is shown in [3, 4] that the Poisson extension with respect to α-stable process has a nontangential limit a.e. on the upper half-space and Lipschitz domains, respectively.

Among many generalizations of the Fatou theorem, it has been proved that, under various types of assumptions on the boundary functions, the nontangential approach can be relaxed (see [12, 31, 32, 36]). The boundedness of modified maximal operators has been an essential tool to prove this type of results. Recently, Mizuta [30] applied analytic tools to the Poisson kernel of $\Delta^{\alpha/2}$ in the half space and showed that under the same assumption as that in

[3], a regular harmonic function (Poisson extension) with respect to α-stable process in the half space has tangential limits a.e. if the approaching region is a certain parabola depending on α.

By using the upper bound of the Poisson kernel for subordinate Brownian motion near the boundary, we show that the regular harmonic function with respect to subordinate Brownian motion, which is the local L^p-Hölder continuous function of order β on D^c with $p \in (1, \infty]$ and $\beta > 1/p$, converges a.e. through a certain parabola. In our results, the approaching region depends on the Laplace exponent of subordinator ϕ and its derivative ϕ'. Nonetheless, our approaching region is always sufficiently wide to contain a Stolz open set. See Remark 4.1.4 to see how wide our approaching region is.

The remainder of this thesis is organized as follows: In Chapter 2, we recall the definition and properties of subordinate Brownian motion and introduce recent results on the potential theory in [20]. Using Green function estimates and behavior of jumping kernel, we obtain the Poisson kernel estimates in Chapter 3. In Chapter 4, we state our main theorem and give a proof using the upper bound of Poisson kernel and boundary Harnack principle.

In this thesis, we use the following convention: We use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$. For a set W in $\mathbb{R}^d$, $\overline{W}$ and W^c denote the closure and complement of W in $\mathbb{R}^d$, respectively. For any open set V, we denote by $\delta_V(x)$, the distance of a point x to the boundary of V, i.e., $\delta_V(x) = \mathrm{dist}(x, \partial V)$. We often denote point $z = (z_1, \dots, z_d) \in \mathbb{R}^d$ as $(\widetilde{z}, z_d)$ with $\widetilde{z} \in \mathbb{R}^{d-1}$. We say that $f\colon \mathbb{R} \to \mathbb{R}$ is increasing if $s \le t$ implies $f(s) \le f(t)$ and analogously for a decreasing function. We use notation $f(t) \asymp g(t)$ as $t \to \infty$ (resp. $t \to 0+$) if the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to \infty$ (resp. $t \to 0+$). The values of the constants $\gamma_1, \gamma_2, C_0^*, C_1^*, C_2^*, C_3^*, C_4^*, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$ will remain the same throughout this thesis, while $c, c_0, c_1, c_2, c_3, \dots$ represent constants whose values are unimportant and may change. All constants are positive finite numbers. The constants $c_0, c_1, c_2, \dots$ are labeled again in the statement and proof of each result. The dependence of constant c on dimension d is not mentioned explicitly. For $x \in \mathbb{R}^d$, $r > 0$, and a set $W \subset \mathbb{R}^d$,

$B_W(x,r) := B(x,r) \cap W$. We denote ω_d the surface area of unit sphere $\partial B(0,1)$ in $\mathbb{R}^d$.

Chapter 2

Preliminaries

2.1 Subordinate Brownian motion

In this section, we introduce certain essential facts about our Lévy process X, which will be used later. First, we review the definition of Lévy process and related basic notions.

An $\mathbb{R}^d$-valued stochastic process $Y = (Y_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if Y has stationary and independent increments, $\mathbb{P}(Y_0 = 0) = 1$, and its sample paths are right continuous with left limits $\mathbb{P}$-a.s. To understand the Lévy process, we often use characteristic exponent Φ, which is given by Fourier transform

$$\mathbb{E}[\exp\{i\zeta \cdot Y_t\}] = \exp\{-t\Phi(\zeta)\}, \quad \zeta \in \mathbb{R}^d.$$

Then, the Lévy-Khintchine formula says the characteristic exponent Φ of Y can be represented as

$$\Phi(\zeta) = i\rho \cdot \zeta + \frac{1}{2}\zeta \cdot Q\zeta + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\zeta \cdot x} + i\zeta \cdot x\, 1_{\{|x|<1\}}\right) \Pi(\mathrm{d}x),$$

where $\rho \in \mathbb{R}^d$, Q is a nonnegative definite $d \times d$ matrix, and $\Pi(\mathrm{d}x)$ is a measure with

$$\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |x|^2)\, \Pi(\mathrm{d}x) < \infty.$$

(Q, ρ, Π) is called the Lévy triple. In particular, Π is called the Lévy measure of Y. If the Lévy measure $\Pi(\mathrm{d}x)$ has a density function $J(x)$, we call it the jumping kernel of Y.

The infinitesimal generator $\mathcal{A}$ of Y is defined by

$$\mathcal{A}u(x) := \lim_{t \downarrow 0} \frac{\{\mathbb{E}_x[u(Y_t)] - u(x)\}}{t}.$$

When the Lévy triple of Y is (Q, ρ, Π), the generator is of the form

$$\begin{aligned}\mathcal{A}u(x) = &-\rho \cdot \nabla u(x) + \frac{1}{2}\sum_{i,j=1}^{d} Q_{ij}\partial_{ij}u(x) \\ &+ \int_{\mathbb{R}^d} \big(u(x+y) - u(x) - \nabla u \cdot y 1_{\{|y|<1\}}\big)\Pi(\mathrm{d}y), \quad u \in \mathcal{C}_c^2.\end{aligned}$$

Let $B = (B_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ whose infinitesimal generator is Δ (our Brownian motion B runs at twice the usual speed), and let $S = (S_t : t \geq 0)$ be a subordinator (increasing Lévy process in $\mathbb{R}$) independent of B whose Laplace exponent is ϕ, i.e.,

$$\mathbb{E}[\exp\{-\lambda S_t\}] = \exp\{-t\phi(\lambda)\}, \qquad \lambda > 0.$$

Then, the process $X = (X_t : t \geq 0)$ defined by $X_t := B_{S_t}$ is a rotationally invariant Lévy process in $\mathbb{R}^d$ and is called the subordinate Brownian motion. The characteristic exponent Φ and the infinitesimal generator $\mathcal{A}$ of X are given by $\Phi(x) = \phi(|x|^2)$ and $\mathcal{A} = \phi(\Delta) := -\phi(-\Delta)$, respectively. It is well-known that the Laplace exponent ϕ of subordinator is always Bernstein function with $\phi(0+) = 0$.

A smooth function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n \phi^{(n)} \leq 0$ for every positive integer n. It is also well-known that every Bernstein function with $\phi(0+) = 0$ has the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(\mathrm{d}t)\,, \quad \lambda > 0\,, \tag{2.1.1}$$

where $b \geq 0$, and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(\mathrm{d}t) < \infty$.

μ is called the Lévy measure of ϕ. (See [33].) If $b = 0$, then the corresponding subordinate Brownian motion is pure jump process.

By concavity, every Bernstein function ϕ satisfies

$$\phi(t\lambda) \leq \lambda\phi(t) \qquad \text{for all } \lambda \geq 1, t > 0. \tag{2.1.2}$$

Thus, $\lambda \mapsto \phi(\lambda)/\lambda$ is decreasing, and therefore,

$$\lambda\phi'(\lambda) \leq \phi(\lambda) \qquad \text{for all } \lambda > 0. \tag{2.1.3}$$

These simple properties of ϕ will be used several times in this thesis.

ajkse

The Lévy measure of X has the density $J(x) = j(|x|)$, where

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\, \mu(\mathrm{d}t)$$

and μ is the Lévy measure of ϕ. Using the jumping kernel, the infinitesimal generator of X can be written by

$$\phi(\Delta)u(x) = b\Delta u(x) + \int_{\mathbb{R}^d} \left(u(x+y) - u(x) - \nabla u(x)\cdot y\mathbf{1}_{\{|y|\leq 1\}}\right) J(y)\,\mathrm{d}y.$$

The probability measure $\mathbb{P}_x(X_t \in \mathrm{d}y)$ has density function $p(t,x,y)$, which is given by

$$p(t,x,y) = \int_0^\infty (4\pi s)^{-d/2} \exp\left(-\frac{|x-y|^2}{4s}\right) \mathbb{P}(S_t \in \mathrm{d}s).$$

We call $p(t,x,y)$ the transition density function (or heat kernel) of X. Recall that X is said to be transient if $\mathbb{P}_0(\lim_{t\to\infty}|X_t| = \infty) = 1$. If X is transient, we can define the Green function $G(x,y)$ by

$$G(x,y) = g(|x-y|) = \int_0^\infty p(t,x,y)\mathrm{d}t.$$

Indeed, $G(x,y)$ is the density function of potential measure V of X, i.e.,

$$\int_A G(x,y)\mathrm{d}y = V(x,A) := \mathbb{E}_x \int_0^\infty \mathbf{1}_A(X_t)\,\mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Using potential measure $\widetilde{V}$ of S, the function g can be written by

$$g(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \widetilde{V}(\mathrm{d}t). \tag{2.1.4}$$

Thus, we see that g is decreasing.

For any open subset U in $\mathbb{R}^d$, let τ_U be the first exit time of U, i.e. $\tau_U = \inf\{t > 0 : X_t \notin U\}$. We use $G_U(x,y)$ to denote the Green function of the process X in U, which can be defined as

$$G_U(x,y) = G(x,y) - \mathbb{E}_x[G(X_{\tau_U}, y)].$$

Then, G_U is the density function of the measure

$$A \mapsto \mathbb{E}_x \int_0^{\tau_U} \mathbf{1}_A(X_t)\,\mathrm{d}t, \quad A \subset U,$$

which says the mean occupation time of X in A before exiting U. For each fixed $z_0 \in U$, the function $G_U(\cdot, z_0)$ is the non-negative regular harmonic function for X in $U \setminus \overline{B(z_0, \epsilon)}$ for every $\epsilon > 0$ and it vanishes on $\mathbb{R}^d \setminus U$.

Now, we define the Poisson kernel by

$$K_U(x,z) := \int_U G_U(x,y) j(|y-z|)\mathrm{d}y, \qquad (x,z) \in U \times \overline{U}^c. \tag{2.1.5}$$

Then, by the result of Ikeda and Watanabe (see [15, Theorem 1]), $K_U(x,z)$ is the density function of $\mathbb{P}_x(X_{\tau_U} \in \mathrm{d}z)$ on $\overline{U}^c$. Thus, for any open subset U and every non-negative measurable function f,

$$\mathbb{E}_x\left[f(X_{\tau_U});\, X_{\tau_U -} \neq X_{\tau_U}\right] = \int_{\overline{U}^c} K_U(x,z) f(z)\mathrm{d}z.$$

2.2 Assumptions on ϕ

In this section, we introduce recent results on potential theory of subordinate Brownian motion, which will be used in Chapter 3 and 4.

In recent paper [20], the following conditions on the Bernstein function ϕ are considered:
(A-1) ϕ is a complete Bernstein function, i.e., the Lévy measure μ of ϕ has a completely monotone density χ, i.e.,

$$(-1)^n \chi^{(n)} \geq 0, \quad n \geq 0.$$

(A-2) $\phi(0+) = 0$ and $\lim_{\lambda\to\infty} \phi(\lambda) = \infty$.
(A-3) There exist constants $\sigma > 0$, $\lambda_0 > 0$, and $\delta \in (0, 1]$ such that

$$\frac{\phi'(\lambda t)}{\phi'(\lambda)} \leq \sigma\, t^{-\delta} \quad \text{for all} \quad t \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0\,.$$

(A-4) If $d \leq 2$, we assume that constant δ in **(A-3)** satisfies $d + 2\delta - 2 > 0$ and that there are $\sigma_0 > 0$ and

$$\delta_0 \in \left(1 - \tfrac{d}{2}, (1 + \tfrac{d}{2}) \wedge (2\delta + \tfrac{d-2}{2})\right)$$

such that

$$\frac{\phi'(\lambda t)}{\phi'(\lambda)} \geq \sigma_0\, t^{-\delta_0} \quad \text{for all} \quad t \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0.$$

(A-5) If the constant δ in **(A-3)** satisfies $0 < \delta \leq \frac{1}{2}$, then we assume that there exist constants $\sigma_1 > 0$ and $\delta_1 \in [\delta, 1)$ such that

$$\frac{\phi(\lambda t)}{\phi(\lambda)} \geq \sigma_1\, t^{1-\delta_1} \quad \text{for all} \quad t \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0\,.$$

(A-6) There exist a $\theta > 0$ such that

$$\int_0^\theta \frac{\lambda^{d/2-1}}{\phi(\lambda)} \mathrm{d}\lambda < \infty.$$

From **(A-3)**, we get $b = 0$ in (2.1.1) by letting $t \to \infty$. From [19, Lemma

2.2], **(A-3)** also implies that for every $\epsilon > 0$, there exists $c = c(\epsilon) > 0$ such that

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \le c x^{1-\delta+\epsilon} \qquad \text{for all } \ x \ge 1 \ \text{ and } \ \lambda \ge \lambda_0. \tag{2.2.1}$$

From the Chung-Fuchs type criterion of the transience of X, **(A-6)** is equivalent to the transience of X (see [19, (2.9)]). Thus, under the assumptions **(A-3)** and **(A-6)**, the process X is a transient pure jump subordinate Brownian motion.

Example 2.2.1. Here are some examples of ϕ that satisfy the above assumptions **(A-1)**–**(A-6)**.

- $\phi(\lambda) = \lambda^{\alpha/2}, \quad \alpha \in (0, 2);$
- $\phi(\lambda) = (\lambda + \lambda^{\alpha})^{\kappa}, \quad \alpha, \kappa \in (0, 1);$
- $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m, \quad \alpha \in (0, 2),\ m > 0,\ d > 2;$
- $\phi(\lambda) = \lambda^{\alpha/2} + \lambda^{\kappa/2}, \quad 0 \le \kappa < \alpha \in (0, 2);$
- $\phi(\lambda) = \lambda^{\alpha/2}(\log(1 + \lambda))^{\kappa}, \quad \alpha \in (0, 2),\ \kappa \in (-\alpha/2, 1 - \alpha/2);$
- $\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \quad \alpha \in (0, 2],\ d > \alpha;$
- $\phi(\lambda) = \log(1 + (\lambda + m^{2/\alpha})^{\alpha/2} - m), \quad \alpha \in (0, 2),\ m > 0,\ d > 2.$

Under the assumptions **(A-1)**–**(A-6)**, $g(r)$ and $j(r)$ enjoy the following estimates (see [19]):

(1) For every $M > 0$, there exists $c = c(M) > 0$ such that

$$c^{-1}\frac{\phi'(r^{-2})}{r^{d+2}\phi(r^{-2})^2} \le g(r) \le c\frac{\phi'(r^{-2})}{r^{d+2}\phi(r^{-2})^2}, \qquad 0 < r \le M, \tag{2.2.2}$$

(2) There exists $C_0^* > 0$ such that

$$j(r) \le C_0^* j(r+1), \qquad \text{for all } \ r > 1. \tag{2.2.3}$$

(3) For every $M > 0$, there exists $C_1^* = C_1^*(M) > 0$ such that

$$(C_1^*)^{-1}\frac{\phi'(r^{-2})}{r^{d+2}} \le j(r) \le C_1^* \frac{\phi'(r^{-2})}{r^{d+2}}, \qquad 0 < r \le M. \tag{2.2.4}$$

Definition 2.2.2. *An open set D in $\mathbb{R}^d$ is said to be $C^{1,1}$ if there exist $R, \Lambda > 0$ such that the following holds: for every $\xi \in \partial D$, there exist*

1. *a $C^{1,1}$-function $\Gamma = \Gamma_\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\Gamma(0) = 0$, $\nabla\Gamma(0) = (0, \dots, 0)$, $\|\nabla\Gamma\|_\infty \le \Lambda$, $|\nabla\Gamma(x) - \nabla\Gamma(w)| \le \Lambda|x - w|$, for $x, w \in \mathbb{R}^{d-1}$ and*

2. *an orthonormal coordinate system $CS_\xi : y = (\widetilde{y}, y_d)$ with origin at ξ such that*

$$B(\xi, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_\xi : y_d > \Gamma(\widetilde{y})\}.$$

The pair (R, Λ) is called the $C^{1,1}$ characteristics of the open set D.

For open set D, we denote $d_D := \mathrm{diam}(D) := \sup\{|x-y| : x, y \in D\}$. The following theorem gives the sharp estimates of Green function in D, when D is a bounded $C^{1,1}$ open set.

Theorem 2.2.3. [20, Theorem 1.2] *Suppose that $X = (X_t, \mathbb{P}_x : t \ge 0, x \in \mathbb{R}^d)$ is a subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, with ϕ satisfying* **(A-1)**–**(A-6)**. *Then, for every bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists $C_2^* = C_2^*(d_D, R, \Lambda, \phi, d) > 1$ such that the Green function $G_D(x, y)$ of X in D satisfies*

$$(C_2^*)^{-1} g_D(x, y) \le G_D(x, y) \le C_2^* g_D(x, y) \tag{2.2.5}$$

with

$$g_D(x, y) = \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}.$$

The following theorem is also one of the main result in [20]. This theorem is called uniform boundary Harnack principle. We will use this theorem in Chapter 4.

Theorem 2.2.4. [20, Theorem 5.6(i)] *Suppose that* ϕ *satisfies* **(A-1)**–**(A-3)**. *There exists a constant* $c = c(\phi) > 0$ *such that the following holds: For every* $z_0 \in \mathbb{R}^d$*, every open set* $D \subset \mathbb{R}^d$*, every* $r \in (0, 1)$*, and for any non-negative functions* u, v *in* $\mathbb{R}^d$*, which are regular harmonic in* $D \cap B(z_0, r)$ *with respect to* X *and vanish a.e. on* $D^c \cap B(z_0, r)$*, we have*

$$\frac{u(x)}{v(x)} \leq c\frac{u(y)}{v(y)}$$

for all $x, y \in D \cap B(z_0, r/2)$.

Chapter 3

Poisson kernel estimates

3.1 Poisson kernel estimates for subordinate Brownian motion

Definition 3.1.1. *An open set $D \subset \mathbb{R}^d$ is said to satisfy cone condition if there exist constants $R > 0$ and $\eta \in (0, 2]$ such that the following holds:*

(1) For any $x \in \overline{D}$, $\overline{\mathcal{C}}(x, R, \eta) \setminus \{x\} \subset D$ for some orthonormal coordinate system CS_x, where $\overline{\mathcal{C}}(x, R, \eta)$ is a closure of $\mathcal{C}(x, R, \eta)$.

(2) For any $z \in \overline{D}^c$ with $\delta_D(z) < R/4$, there exist $z_0 \in \partial D$ such that $\delta_D(z) \le |z - z_0| \le 2\delta_D(z)$ and corresponding cone $\mathcal{C}(z_0, R, \eta)$ which is contained in D for some coordinate system CS_{z_0}. In particular $\tilde{z} = \tilde{0}$ in CS_{z_0}.

The pair (R, η) is called cone characteristic constant of the open set D.

Note that Lipschitz open set (see Definition 4.2.1) satisfies the above cone condition.

The following theorem is the main result in this Chapter.

Theorem 3.1.2. *Suppose $M > 0$ and that $X = (X_t : t \ge 0)$ is a Lévy process whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where $\phi : (0, \infty) \to [0, \infty)$ is a Bernstein function with $\phi(0+) = 0$ and*

$\lim_{t\to\infty}\phi(t) = \infty$. *We assume that there exists a increasing function* $\psi : ((5M)^{-2}, \infty) \to (0,\infty)$ *and a constant* $c_1 \geq 1$ *such that*

$$c_1^{-1}\psi(\lambda) \leq \lambda^{1+d/2}\phi'(\lambda)/\phi(\lambda) \leq c_1\psi(\lambda), \quad \lambda \in ((5M)^{-2}, \infty). \tag{3.1.1}$$

Then (2.2.3), (2.2.4), *and* (2.2.5) *imply that if a bounded open set* D *satisfies the cone condition with cone characteristic constant* (R,η) *and* $d_D < M$, *then there exists* $c = c(c_1, C_0^*, C_1^*, C_2^*, R/d_D, \eta, M, d) > 1$ *such that*

$$\begin{aligned} &c^{-1}\frac{\phi(\delta_D(z)^{-2})^{1/2}j(|x-z|)}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(d_D^{-2})^{1/2}\phi(\delta_D(z)^{-2})^{-1/2})} \\ &\leq K_D(x,z) \leq c\,\frac{\phi(\delta_D(z)^{-2})^{1/2}j(|x-z|)}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(d_D^{-2})^{1/2}\phi(\delta_D(z)^{-2})^{-1/2})}, \end{aligned} \tag{3.1.2}$$

where C_0^*, C_1^* *and* C_2^* *are constants satisfying* (2.2.3), (2.2.4), *and* (2.2.5).

The assumption (3.1.1) is very mild. For example, if ϕ is a special Bernstein function ($\lambda \mapsto \lambda/\phi(\lambda)$ is a Bernstein function), then $\lambda \mapsto \lambda^2\phi'(\lambda)/\phi(\lambda)^2$ is increasing for all $\lambda > 0$ (see [19, Lemma 3.1]). Moreover, if $G(x,y) = g(|x-y|) \asymp \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}$ as $|x-y| \to 0$, then (3.1.1) is always true because $g(\lambda)$ is decreasing. Note that the term $1+\phi(d_D^{-2})^{1/2}\phi(\delta_D(z)^{-2})^{-1/2}$ appears in (3.1.2) since the constant c in Theorem 3.1.2 depends on R/d_D, but neither on R nor d_D.

Although (3.1.2) follows from integration and estimation, due to our general formulation, it is not straightforward. Nevertheless, assumptions on the set D are mild; it may be just a bounded Lipschitz or $C^{1,\beta}$ open set for some $\beta \in (0,1)$. It is worth mentioning that the constant c in Theorem 3.1.2 depends on R/d_D, thereby allowing uniform estimates of Poisson kernels of balls with constant not depending on the radii of balls (cf. Corollary 3.2.7).

Due to [19, 20], under the assumptions **(A-1)**–**(A-6)**, (2.2.3)–(2.2.5) hold and

$$G(x,y) = g(|x-y|) \asymp \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}$$

as $|x-y| \to 0$ so that (3.1.1) also holds. Therefore, applying Theorem 3.1.2, we have the sharp two-sided estimates for Poisson kernel for a large class of subordinate Brownian motion including geometric stable process.

Theorem 3.1.3. *Suppose that $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$ satisfying* **(A-1)**–**(A-6)**. *Then, for every bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists $c = c(d_D, R, \Lambda, \phi, d) > 1$ such that*

$$
\begin{aligned}
&c^{-1}\frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(\delta_D(z)^{-2})^{-1/2})}j(|x-z|)\\
&\leq K_D(x,z) \leq c\,\frac{\phi(\delta_D(z)^{-2})^{1/2}}{\phi(\delta_D(x)^{-2})^{1/2}\phi(|x-z|^{-2})(1+\phi(\delta_D(z)^{-2})^{-1/2})}j(|x-z|).
\end{aligned}
$$

Example 3.1.4. When the subordinator has the Laplace exponent

$$\phi(\lambda) = \log(1+\lambda^{\alpha/2}) \qquad (0 < \alpha \leq 2, d > \alpha),$$

by [24, Lemma 3.3] and our Theorem 3.1.3, we have

$$
\begin{aligned}
&K_D(x,z)\\
&\asymp \begin{cases} \frac{(\log(1+\delta_D(z)^{-\alpha}))^{1/2}}{(\log(1+\delta_D(x)^{-\alpha}))^{1/2}(1+(\log(1+\delta_D(z)^{-\alpha}))^{-1/2})}\frac{|x-z|^{-d}}{(\log(1+|x-z|^{-\alpha}))^{1/2}}, & \delta_D(z) \leq 2d_D\\ \frac{\delta_D(x)^{\alpha/2}}{\delta_D(z)^{\alpha/2}(1+\delta_D(z)^{\alpha/2})}|x-z|^{-d}, & \delta_D(z) > 2d_D. \end{cases}
\end{aligned}
$$

Note that when $\phi(\lambda) = \lambda^{\alpha/2}$, it is known that

$$K_D(x,z) \asymp \frac{\delta_D(x)^{\alpha/2}}{\delta_D(z)^{\alpha/2}(1+\delta_D(z)^{\alpha/2})}|x-z|^{-d}.$$

(See [10, 26].)

3.2 Proof of Poisson kernel estimates

In order to cover more general Lévy processes, we give the proof under slightly weaker assumptions. Throughout this section, D is a bounded open set with $d_D < M$ for some $M \geq 1$.

We assume the function $\Phi : [0, \infty) \to [0, \infty)$ satisfies the following properties:

(P1) Φ is an increasing C^1-function with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

(P2) There exists a constant $C_0 \geq 1$ such that

$$\Phi(t\lambda) \leq C_0 \lambda^2 \Phi(t) \qquad \text{for all } \lambda \geq 1, t > 0. \tag{3.2.1}$$

(P3) There exists a constant $C_1 > 0$ such that

$$\Phi'(t\lambda) \leq C_1 \lambda \Phi'(t) \qquad \text{for all } \lambda \geq 1, t > 0. \tag{3.2.2}$$

(P4) There exists an increasing function $\Psi : ((5M)^{-1}, \infty) \to (0, \infty)$ and a constant $C_2 \geq 1$ such that

$$C_2^{-1}\Psi(\lambda) \leq \lambda^{1+d} \frac{\Phi'(\lambda)}{\Phi(\lambda)} \leq C_2 \Psi(\lambda), \quad \lambda \in ((5M)^{-1}, \infty).$$

We assume $X := (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a purely discontinuous symmetric transient Lévy process such that the characteristic exponent of X is $\Phi_X(\zeta)$ and the Lévy measure of X has a density $J(x)$ and $\mathbb{P}_x(X_0 = x) = 1$. Then

$$\mathbb{E}_x \left[e^{i\zeta\cdot(X_t - X_0)} \right] = e^{-t\Phi_X(\zeta)}, \quad x \text{ and } \zeta \in \mathbb{R}^d,$$

with

$$\Phi_X(\zeta) = \int_{\mathbb{R}^d} (1 - \cos(\zeta \cdot y)) J(y) \mathrm{d}y.$$

We further assume that

(J1) There exist a decreasing function $j : (0, \infty) \to (0, \infty)$ and constants $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 j(|x|) \leq J(x) \leq \gamma_2 j(|x|). \tag{3.2.3}$$

We assume that the Green function $G_D(x, y)$ and the function j in **(J1)** satisfies the following estimates:

(G) There exist positive constants C_3 and C_4 such that

$$\begin{aligned} &C_3 \left(1 \wedge \tfrac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2} \left(1 \wedge \tfrac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \tfrac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \\ &\le G_D(x,y) \\ &\le C_4 \left(1 \wedge \tfrac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2} \left(1 \wedge \tfrac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \tfrac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}\,. \end{aligned} \tag{3.2.4}$$

(J2) There exist positive constants $C_5 = C_5(M)$ and $C_6 = C_6(M)$ such that

$$C_5 \frac{\Phi'(r^{-1})}{r^{d+1}} \le j(r) \le C_6 \frac{\Phi'(r^{-1})}{r^{d+1}}, \qquad r \in (0, 10M). \tag{3.2.5}$$

(J3) There exists $C_7 > 0$ such that

$$j(r) \le C_7 j(r+1), \qquad r > 1. \tag{3.2.6}$$

Note that **(P3)** and **(J2)** imply that there exists $C_8 > 0$ such that

$$j(r) \le C_8 j(2r), \qquad r \in (0, 5M). \tag{3.2.7}$$

In fact,

$$j(r) \le C_6 \frac{\Phi'(r^{-1})}{r^{d+1}} \;\le\; 2C_1C_6 \frac{\Phi'(2^{-1}r^{-1})}{r^{d+1}} \;\le\; C_1C_5^{-1}C_6 2^{d+2} j(2r), \quad r \in (0, 5M).$$

Also, by using the assumption that Φ is increasing and (3.2.1), it follows that (3.2.4) is equivalent to

$$\begin{aligned} &C_3^* \left(1 \wedge \tfrac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}}\right) \tfrac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \\ &\le\; G_D(x,y) \;\le\; C_4^* \left(1 \wedge \tfrac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}}\right) \tfrac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \end{aligned} \tag{3.2.8}$$

for some positive constant C_3^*, C_4^*. Indeed,

$$\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right) \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right) \le 1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi(\delta_D(x)^{-1})\Phi(\delta_D(y)^{-1})}.$$

Since other cases are similar or easy to check, we will show that

$$1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi(\delta_D(x)^{-1})\Phi(\delta_D(y)^{-1})} \le 4C_0 \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right) \tag{3.2.9}$$

when $\delta_D(y) \le |x-y| \le \delta_D(x)$. In this case, $\delta_D(x) \le \delta_D(y)+|x-y| \le 2|x-y|$. Thus

$$\begin{aligned} 1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi(\delta_D(x)^{-1})\Phi(\delta_D(y)^{-1})} &\le 1 \wedge \frac{\Phi(|x-y|^{-1})^2}{\Phi((2|x-y|)^{-1})\Phi(\delta_D(y)^{-1})} \\ &\le 1 \wedge \frac{4C_0\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})} \\ &\le 4C_0\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right), \end{aligned}$$

which implies (3.2.9). This shows that (3.2.8) is equivalent to (3.2.4).

As in (2.1.5) we denote the Poisson kernel of X in $D \times \overline{D}^c$ by $K_D(x,z)$.

Remark 3.2.1. When Φ is of the form $\Phi(\lambda) = \phi(\lambda^2)$, we can check **(P1)**–**(P4)** for some particular cases of ϕ:

(1) ϕ is a Bernstein function with $\phi(0+) = 0$:

In this case, Φ is increasing C^∞-function and $\Phi'(\lambda) = 2\lambda\phi'(\lambda^2)$. By concavity, every Bernstein function ϕ satisfies $\phi(t\lambda) \le \lambda\phi(t)$ for all $\lambda \ge 1, t > 0$. So we have **(P2)** with $C_0 = 1$. Since ϕ' is decreasing, we have **(P3)** with $C_1 = 1/2$. So, for a Bernstein function ϕ, **(P2)** and **(P3)** hold. If ϕ has further property such that $\lim_{t\to\infty}\phi(t) = \infty$ then $\lim_{t\to\infty}\Phi(t) = \infty$ which implies **(P1)**. In fact, $\lim_{t\to\infty}\phi(t) = \infty$ holds when Lévy measure of X is infinite.

(2) ϕ is a special Bernstein function, i.e. $\lambda \mapsto \frac{\lambda}{\phi(\lambda)}$ is also a Bernstein function:

By [19, Lemma 3.1], $\lambda \mapsto \lambda^2\phi'(\lambda)/\phi(\lambda)^2$ is increasing for all $\lambda > 0$. Since $\lambda^{1+d}\Phi'(\lambda)/\Phi(\lambda) = 2(\lambda^2)^{1+d/2}\phi'(\lambda^2)/\phi(\lambda^2)$ and ϕ is increasing, (**P4**) holds if $d \ge 2$. Thus for a special Bernstein function **(P4)** holds for $d \ge 2$. Note that **(P2)** and **(P3)** also hold by (1).

(3) ϕ is a Laplace exponent of subordinator which satisfies the assumption **(A-1)**–**(A-3)** and **(B)** in [19]:

In this case, Lévy process X is a subordinate Brownian motion with Lévy exponent Φ and ϕ is of the form (2.1.1) with $\phi(0) = 0$ ($b = 0$) and $\lim_{t\to\infty} \phi(t) = \infty$. Hence **(P1)**, **(P2)** and **(P3)** hold. By [19, Proposition 4.2], we get **(J2)** and if X is transient then by [19, Proposition 4.5], $g(r) \asymp r^{-2-d}\phi'(r^{-2})/\phi(r^{-2})^2$ as $r \to 0+$ which implies **(P4)** holds. In fact, [19, Remark 3.1(i)] says ϕ is a special Bernstein function. So, we have **(P4)** for $d \geq 2$ without **(B)** and transience of X.

(4) ϕ is a Laplace exponent of subordinator which satisfies assumptions **(A-1)**–**(A-5)**:

(J1), **(J2)** and **(J3)** hold by [20, Proposition 2.6] and statements following it. Since ϕ is a Bernstein function which is of the form (2.1.1) and satisfies **(A-2)**, it can be seen as in (3) that **(P1)**, **(P2)**, **(P3)** hold. When X is transient, we have **(G)** by Theorem 2.2.3 and $g(\lambda^{-1}) \asymp \lambda^{2+d}\phi'(\lambda^2)/\phi(\lambda^2)^2$ which implies **(P4)** since $g(r)$ is decreasing.

It follows from Remark 3.2.1 that if ϕ satisfies the assumptions in Theorem 3.1.2, then $\Phi(\lambda) = \Phi_X(\lambda) = \phi(\lambda^2)$ satisfies **(P1)**–**(P4)** and (2.2.5), (2.2.3), (2.2.4) imply **(G)**, **(J1)**, **(J2)** and **(J3)**. For the remainder of this section we assume that Φ satisfies **(P1)**–**(P4)**. We want to estimate $K_D(x, z)$ in terms of Φ when **(G)**, **(J1)**, **(J2)** and **(J3)** hold.

We first consider the case $\delta_D(z) > 2d_D$.

Proposition 3.2.2. *If* (3.2.3), (3.2.6) *and* (3.2.7) *hold, then there exist* $c_1 = c_1(\gamma_1, C_7, C_8, M) > 0$ $c_2 = c_2(\gamma_2, C_7, C_8, M) > 0$ *such that for* $z \in \overline{D}^c$ *with* $\delta_D(z) > 2d_D$

$$c_1 \int_D G_D(x, y)\mathrm{d}y\, j(|x - z|) \leq K_D(x, z) \leq c_2 \int_D G_D(x, y)\mathrm{d}y\, j(|x - z|). \tag{3.2.10}$$

In addition, if the upper bound of $G_D(x,y)$ in (3.2.4) *holds then there exists* $c_3 = c_3(\gamma_2, C_4, C_7, C_8, d, M) > 0$ *such that for* $z \in \overline{D}^c$ *with* $\delta_D(z) > 2d_D$

$$K_D(x,z) \le c_3 \frac{j(|x-z|)}{\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(x)^{-1})^{1/2}}. \tag{3.2.11}$$

Proof. We note that

$$|y-z| - d_D \le |y-z| - |x-y| \le |x-z| \le |y-z| + |x-y| \le |y-z| + d_D. \tag{3.2.12}$$

We consider two cases: $2d_D < \delta_D(z) \le 2M$ and $\delta_D(z) > 2M$ separately to prove (3.2.10). First, consider the case $2d_D < \delta_D(z) \le 2M$. Since $|y-z| > 2d_D$, by (3.2.12) we have

$$\frac{1}{2}|y-z| < |x-z| < \frac{3}{2}|y-z|.$$

Since $|x-z|, |y-z| \le 2M + d_D < 3M$, (3.2.10) follows from (3.2.3) and (3.2.7) in this case. If $\delta_D(z) > 2M$, then $2M < |y-z|$. Since $|y-z| - d_D < |x-z| < |y-z| + d_D$ and $d_D < M$, we have

$$|y-z| - M < |x-z| < |y-z| + M.$$

This, (3.2.3) and (3.2.6) prove (3.2.10) since $|y-z| - M > M \ge 1$. Hence for $\delta_D(z) > 2d_D$, (3.2.10) holds.

Now we further assume that the upper bound of $G_D(x,y)$ in (3.2.4) holds.

Then,

$$\begin{aligned}\int_D G_D(x,y)\mathrm{d}y &\le C_4 \int_D \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2} \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \\ &\qquad \times \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}\mathrm{d}y \\ &\le C_4 \int_D \frac{\Phi'(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}}\mathrm{d}y \\ &\le \frac{C_4\omega_d}{\Phi(\delta_D(x)^{-1})^{1/2}} \int_0^{d_D} 2(\Phi(r^{-1})^{-1/2})'\mathrm{d}r \\ &= \frac{2C_4\omega_d}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}.\end{aligned}$$

In the last equality, we have used $\lim_{t\to\infty}\Phi(t)=\infty$. □

We now give the upper bound of $K_D(x,z)$ when $\delta_D(z) \le 2d_D$.

Proposition 3.2.3. *Assume* (3.2.3) *and suppose that the upper bounds of* $G_D(x,y)$ *and* $j(|x|)$ *are given by* (3.2.4) *and* (3.2.5), *respectively. Then there exists* $c = c(\gamma_2, C_0, C_1, C_2, C_4, C_6, d) > 0$ *such that for every* $x \in D$ *and* $z \in \overline{D}^c$ *with* $\delta_D(z) \le 2d_D$,

$$K_D(x,z) \le c\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}.$$

Proof. By (2.1.5), we have

$$\begin{aligned}K_D(x,z) &= \int_D G_D(x,y)J(y-z)\mathrm{d}y \\ &= \int_{\{y\in D:|x-z|<2|x-y|\}} G_D(x,y)J(y-z)\mathrm{d}y \\ &\qquad + \int_{\{y\in D:|x-z|\ge 2|x-y|\}} G_D(x,y)J(y-z)\mathrm{d}y =: I + II.\end{aligned}$$

By (3.2.4) we have the following estimate:

$$G_D(x,y) \le \frac{C_4\Phi'(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}|x-y|^{d+1}\Phi(|x-y|^{-1})}. \tag{3.2.13}$$

$$G_D(x,y) \le \frac{C_4\Phi'(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}}. \tag{3.2.14}$$

When $|x-z| < 2|x-y|$, by using **(P4)**, (3.2.2) and the assumption that Φ is increasing,

$$\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})} \le \frac{C_2^2 2^{d+1}\Phi'(2|x-z|^{-1})}{|x-z|^{d+1}\Phi(2|x-z|^{-1})} \le \frac{c_1\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})} \tag{3.2.15}$$

where $c_1 = C_1C_2^2 2^{d+2}$. Since $|y-z| \le 3d_D < 3M$, by (3.2.5),

$$j(|y-z|) \le C_6\Phi'(|y-z|^{-1})/|y-z|^{d+1}$$

holds. Using this, (3.2.3), (3.2.13), (3.2.15) and polar coordinates,

$$\begin{aligned}
I \le& \frac{\gamma_2C_4c_1C_6}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})} \\
&\times\int_{\{y\in D:|x-z|<2|x-y|\}}\frac{1}{\Phi(\delta_D(y)^{-1})^{1/2}}\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}\mathrm{d}y \\
\le& \frac{\gamma_2C_4c_1C_6}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\int_D\frac{1}{\Phi(|y-z|^{-1})^{1/2}}\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}\mathrm{d}y \\
\le& \frac{\gamma_2C_4c_1C_6\omega_d}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\int_{\delta_D(z)}^{\delta_D(z)+d_D}\frac{1}{\Phi(r^{-1})^{1/2}}\frac{\Phi'(r^{-1})}{r^{d+1}}r^{d-1}\mathrm{d}r \\
\le& \frac{2\gamma_2C_4c_1C_6\omega_d}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\int_{\delta_D(z)}^{\infty}-(\Phi(r^{-1})^{1/2})'\mathrm{d}r \\
\le& 2\gamma_2C_4c_1C_6\omega_d\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}.
\end{aligned}$$

The second inequality follows from the fact that $\delta_D(y) \le |y-z|$ and the last inequality follows from $\Phi(0) = 0$.

On the other hand, when $|x-z| \geq 2|x-y|$, we have

$$|y-z| \geq |x-z| - |x-y| \geq \frac{1}{2}|x-z| \geq |x-y|. \tag{3.2.16}$$

Thus by using **(P4)**, (3.2.2) and the assumption that Φ is increasing,

$$\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}} \leq c_1 \Phi(|y-z|^{-1}) \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})} \tag{3.2.17}$$

as in (3.2.15). From (3.2.3), (3.2.5), (3.2.14) and (3.2.17), we get

$$\begin{aligned} II \leq & \gamma_2 C_4 c_1 C_6 \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}} \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})} \\ & \times \int_{\{y \in D: |x-z| \geq 2|x-y|\}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} \Phi(|y-z|^{-1}) \mathrm{d}y. \end{aligned} \tag{3.2.18}$$

Let $a := |x-z|$. By the triangle inequality and (3.2.16),

$$\begin{aligned} & \int_{\{y \in D: |x-z| \geq 2|x-y|\}} \frac{\Phi'(|x-y|^{-1})\Phi(|y-z|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} \mathrm{d}y \\ \leq & \int_{\{y \in D: |x-z| \geq 2|x-y|\}} \frac{\Phi'(|x-y|^{-1})\Phi((||x-z|-|x-y||)^{-1} \wedge |x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} \mathrm{d}y \\ \leq & \, \omega_d \int_0^{d_D} \frac{\Phi'(r^{-1})}{r^{d+1}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1} \wedge r^{-1}) r^{d-1} \mathrm{d}r \\ = & \, \omega_d \int_0^{d_D} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1} \wedge r^{-1}) \mathrm{d}r. \end{aligned}$$

We split the above integral as

$$\begin{aligned} & \int_0^{d_D} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1} \wedge r^{-1}) \mathrm{d}r \\ \leq & \int_0^{\frac{a}{2}} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(|a-r|^{-1}) \mathrm{d}r + \int_{\frac{a}{2}}^{\infty} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \Phi(r^{-1}) \mathrm{d}r \\ \leq & \, \Phi(2a^{-1}) \int_0^{\frac{a}{2}} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{3/2}} \mathrm{d}r + \int_{\frac{a}{2}}^{\infty} \frac{\Phi'(r^{-1})}{r^{2}\Phi(r^{-1})^{1/2}} \mathrm{d}r. \end{aligned}$$

By using $\lim_{t\to\infty}\Phi(t)=\infty$ and $\Phi(0)=0$ respectively, we have

$$\int_0^{\frac{a}{2}}\frac{\Phi'(r^{-1})}{r^2\Phi(r^{-1})^{3/2}}\mathrm{d}r=2\int_0^{\frac{a}{2}}(\Phi(r^{-1})^{-1/2})'\mathrm{d}r=2\Phi(2a^{-1})^{-1/2}$$

and

$$\int_{\frac{a}{2}}^{\infty}\frac{\Phi'(r^{-1})}{r^2\Phi(r^{-1})^{1/2}}\mathrm{d}r=2\int_{\frac{a}{2}}^{\infty}-(\Phi(r^{-1})^{1/2})'\mathrm{d}r=2\Phi(2a^{-1})^{1/2}.$$

Thus by using **(P2)**,

$$\begin{aligned}&\int_{\{y\in D:|x-z|\geq 2|x-y|\}}\frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}}\Phi(|y-z|^{-1})\mathrm{d}y\\&\leq 4\omega_d\Phi(2|x-z|^{-1})^{1/2}\leq 8\omega_dC_0^{1/2}\Phi(|x-z|^{-1})^{1/2}\leq 8\omega_dC_0^{1/2}\Phi(\delta_D(z)^{-1})^{1/2}.\end{aligned}$$

Combining this with (3.2.18), we have

$$II\leq 8c_1C_0^{1/2}\gamma_2C_4C_6\omega_d\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}.$$

Thus

$$K_D(x,z)=I+II\leq c\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}$$

for some $c=c(\gamma_2,C_0,C_1,C_2,C_4,C_6,d)>0$. This finishes the proof. □

Note that in Proposition 3.2.3 we do not need the cone condition of D. In the remainder of this chapter, we assume further that the bounded open set D satisfies the cone condition with cone characteristic constant (R,η) (cf. Definition 3.1.1).

Proposition 3.2.4. *Suppose that* (3.2.3), (3.2.6) *and* (3.2.7) *hold and that the lower bound of* $G_D(x,y)$ *in* (3.2.4) *holds. Then there exists* $c=c(\gamma_1,C_0,C_3,C_7,C_8,R/d_D,\eta,M,d)>0$ *such that for* $z\in\overline{D}^c$ *with* $\delta_D(z)>2d_D$

$$K_D(x,z)\geq c\frac{j(|x-z|)}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}.$$

Proof. By (3.2.10), we only need to show that

$$
\begin{aligned}
h(x) := &\int_D \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2} \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \\
&\qquad \times \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \mathrm{d}y \\
\geq &\frac{c}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}. \qquad (3.2.19)
\end{aligned}
$$

Since D satisfies the cone condition and $x \in D$, there exists a cone $\mathcal{C}(x, R, \eta) \subset D$ for some coordinate system CS_x. So, $E_x := \mathcal{C}(x, R, \eta/2)$ is also in D in the same coordinate system CS_x. Then there exists a constant $c_1 = c_1(\eta) \in (0, 1]$ such that $c_1|x-y| \leq \delta_D(y)$ for $y \in E_x$. This and (3.2.1) imply that $\Phi(\delta_D(y)^{-1})^{1/2} \leq C_0^{1/2} c_1^{-1} \Phi(|x-y|^{-1})^{1/2}$ for $y \in E_x$. Let $c_2 = C_0^{1/2} c_1^{-1} \geq 1$. Since $\delta_D(x) < d_D$ and $|x-y| \leq d_D$ for all $y \in D$, on E_x we have

$$
\begin{aligned}
&\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2} \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \\
= &\frac{(\Phi(\delta_D(x)^{-1}) \wedge \Phi(|x-y|^{-1}))^{1/2}}{\Phi(\delta_D(x)^{-1})} \left(\Phi(\delta_D(x)^{-1}) \wedge \frac{\Phi(\delta_D(x)^{-1})\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2} \\
\geq &\frac{1}{\Phi(\delta_D(x)^{-1})} \Phi(d_D^{-1})^{1/2} (\Phi(d_D^{-1})/c_2)^{1/2}.
\end{aligned}
$$

Thus using (3.2.1) with $c_3 = c_2^{1/2}$, we get

$$
\begin{aligned}
h(x) \geq &\frac{\Phi(d_D^{-1})}{c_3 \Phi(\delta_D(x)^{-1})} \int_{E_x} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2} \mathrm{d}y \\
\geq &\frac{c_4 \omega_d \Phi(d_D^{-1})}{c_3 \Phi(\delta_D(x)^{-1})} \int_0^R \frac{\Phi'(r^{-1})}{r^2 \Phi(r^{-1})^2} \mathrm{d}r = \frac{c_4 \omega_d \Phi(d_D^{-1})}{c_3 \Phi(\delta_D(x)^{-1})} \int_0^R (1/\Phi(r^{-1}))' \mathrm{d}r \\
= &\frac{c_4 \omega_d \Phi(d_D^{-1})}{c_3 \Phi(\delta_D(x)^{-1}) \Phi(R^{-1})} \geq \frac{c_4 \omega_d (R/d_D)^2}{c_3 C_0 \Phi(\delta_D(x)^{-1})} \qquad (3.2.20)
\end{aligned}
$$

for some $c_4 = c_4(\eta) > 0$.

Take $c_5 = R/(4d_D)$ and define $V_x := \{y \in \mathcal{C}(x, R, \eta/2) : c_5 \delta_D(x) <$

$|x-y|\}$. Note that $2c_5\delta_D(x) < R$ since $\delta_D(x) < d_D$. So, for $y \in V_x$, $C_0^{1/2}c_5^{-1}\Phi(\delta_D(x)^{-1})^{1/2} \geq \Phi(|x-y|^{-1})^{1/2}$. Since $V_x \subset E_x$, $\Phi(\delta_D(y)^{-1})^{1/2} \leq C_0^{1/2}c_1^{-1}\Phi(|x-y|^{-1})^{1/2}$ for $y \in V_x$. From these facts, for some $c_6 = c_6(\eta) > 0$, we have

$$\begin{aligned}
h(x) \geq &\frac{c_1 c_5}{C_0}\int_{V_x} \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^{3/2}} \mathrm{d}y \\
\geq &\frac{c_1 c_5 c_6 \omega_d}{C_0 \Phi(\delta_D(x)^{-1})^{1/2}} \int_{c_5\delta_D(x)}^{R} \frac{\Phi'(r^{-1})}{r^2\Phi(r^{-1})^{3/2}} \mathrm{d}r \\
= &\frac{2c_1 c_5 c_6 \omega_d}{C_0 \Phi(\delta_D(x)^{-1})^{1/2}} \int_{c_5\delta_D(x)}^{R} (\Phi(r^{-1})^{-1/2})' \mathrm{d}r \\
= &\frac{2c_1 c_5 c_6 \omega_d}{C_0 \Phi(\delta_D(x)^{-1})^{1/2}} \left(\frac{1}{\Phi(R^{-1})^{1/2}} - \frac{1}{\Phi(c_5^{-1}\delta_D(x)^{-1})^{1/2}} \right).
\end{aligned} \tag{3.2.21}$$

Let $c_7 := c_4\omega_d 2^{-1}c_3^{-1}C_0^{-1}(R/d_D)^2$ and choose $c_8 := c_1c_5c_6\omega_d C_0{}^{-1} \wedge c_7$. Then by (3.2.20) and (3.2.21)

$$\begin{aligned}
h(x) = &\frac{1}{2}h(x) + \frac{1}{2}h(x) \\
\geq &\frac{c_7}{\Phi(\delta_D(x)^{-1})} + \frac{c_8}{\Phi(\delta_D(x)^{-1})^{1/2}} \left(\frac{1}{\Phi(R^{-1})^{1/2}} - \frac{1}{\Phi(c_5^{-1}\delta_D(x)^{-1})^{1/2}} \right) \\
= &\frac{c_8}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(R^{-1})^{1/2}} \\
&\quad + \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}} \left(\frac{c_7}{\Phi(\delta_D(x)^{-1})^{1/2}} - \frac{c_8}{\Phi(c_5^{-1}\delta_D(x)^{-1})^{1/2}} \right) \\
\geq &\frac{c_8}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(R^{-1})^{1/2}} \;\geq\; \frac{c_8 R}{C_0^{1/2} d_D \Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}.
\end{aligned}$$

The penultimate inequality follows from the facts that $c_5 < 1$ and Φ is increasing. The claim (3.2.19) is proved. □

Proposition 3.2.5. *Assume* (3.2.3) *and suppose that the lower bounds of* $G_D(x,y)$ *and* $j(|x|)$ *are given by* (3.2.4) *and* (3.2.5), *respectively. Then there exists* $c = c(\gamma_1, C_0, C_1, C_2, C_3, C_5, \eta, R/d_D, d) > 0$ *such that for every* $x \in D$

and $z \in \overline{D}^c$ *with* $\delta_D(z) \le 2d_D$,

$$K_D(x,z) \ge c\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})^{1/2}}.$$

Proof. Since $|x-z| \ge \delta_D(x)$ and Φ is increasing, we have

$$\begin{aligned}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right) =&\frac{\Phi(|x-z|^{-1})}{\Phi(\delta_D(x)^{-1})}\left(\frac{\Phi(\delta_D(x)^{-1})}{\Phi(|x-z|^{-1})} \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(|x-z|^{-1})}\right)\\ \ge&\frac{\Phi(|x-z|^{-1})}{\Phi(\delta_D(x)^{-1})}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(|x-z|^{-1})}\right).\end{aligned}$$

Thus, by (3.2.3), (3.2.4) and (3.2.5), there exists a constant $c_1 = c_1(\gamma_1, C_3, C_5)$ such that

$$\begin{aligned}K_D(x,z) \ge c_1 &\int_D \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(x)^{-1})}\right)^{1/2}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2}\\ &\qquad\times \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2}\frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}\mathrm{d}y\\ \ge c_1 &\frac{\Phi(|x-z|^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}|x-z|^d}A(x,z),\end{aligned} \tag{3.2.22}$$

where

$$\begin{aligned}A(x,z) := &\int_D \left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(|x-z|^{-1})}\right)^{1/2}\left(1 \wedge \frac{\Phi(|x-y|^{-1})}{\Phi(\delta_D(y)^{-1})}\right)^{1/2}\\ &\qquad\times \frac{\Phi'(|x-y|^{-1})\Phi'(|y-z|^{-1})|x-z|^d}{|x-y|^{d+1}\Phi(|x-y|^{-1})^2|y-z|^{d+1}}\mathrm{d}y.\end{aligned}$$

Let $a = |x-z|$ and $D_a := a^{-1}(D-x)$. Note that $0 \in D_a$ and $(3d_D)^{-1} < a^{-1} < \infty$. By change of variable $y - x = |x-z|\hat{y}$ and using the triangle inequality $|y-z| \le |x-z|+|y-x| = (1+|\hat{y}|)|x-z| < 4M$, we have $|y-x|^{-1} = a^{-1}|\hat{y}|^{-1}$ and $|y-z|^{-1} \ge a^{-1}(1+|\hat{y}|)^{-1} > (4M)^{-1}$. Also, $\delta_D(y) = a\delta_{D_a}(\hat{y})$ where

$\delta_{D_a}(\hat{y}) = \mathrm{dist}(\hat{y}, \partial D_a)$. Then,

$$\begin{aligned} C_2^2 \frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}} &\geq \Phi(|y-z|^{-1}) \frac{\Phi'(a^{-1}(1+|\hat{y}|)^{-1})}{a^{d+1}(1+|\hat{y}|)^{d+1}\Phi(a^{-1}(1+|\hat{y}|)^{-1})} \\ &\geq \frac{\Phi'(a^{-1}(1+|\hat{y}|)^{-1})}{a^{d+1}(1+|\hat{y}|)^{d+1}}, \end{aligned}$$

where the first inequality follows from **(P4)** and the second inequality holds since Φ is increasing. This implies that

$$\begin{aligned} A(x,z) \; \geq \; a^{-2}C_2^{-2} \int_{D_a} & \frac{\Phi'(a^{-1}(1+|\hat{y}|)^{-1})}{(1+|\hat{y}|)^{d+1}} \frac{\Phi'(a^{-1}|\hat{y}|^{-1})}{\Phi(a^{-1}|\hat{y}|^{-1})^2|\hat{y}|^{d+1}} \\ & \times \Big(1 \wedge \frac{\Phi(a^{-1}|\hat{y}|^{-1})}{\Phi(a^{-1})}\Big)^{1/2} \Big(1 \wedge \frac{\Phi(a^{-1}|\hat{y}|^{-1})}{\Phi(a^{-1}\delta_{D_a}(\hat{y})^{-1})}\Big)^{1/2} \mathrm{d}\hat{y}. \end{aligned} \tag{3.2.23}$$

Since D satisfies the cone condition with cone characteristics (R,η), there is a cone $\mathcal{C}(w,R,\eta) \subset D$ for all $w \in D$. So $\hat{\mathcal{C}}(0,R/a,\eta) = a^{-1}(\mathcal{C}(x,R,\eta) - x) \subset D_a$. Since $a \leq 3d_D$, we have $\hat{\mathcal{C}}(0,R/3d_D,\eta) \subset D_a$. By taking $r_1 = R/3d_D \leq 1/3$, we have $P := \hat{\mathcal{C}}(0,r_1,\eta/2) \subset D_a$ in some coordinate system CS_0. Then there exists $c_2 = c_2(\eta) \in (0,1]$ such that $c_2|\hat{y}| \leq \delta_{D_a}(\hat{y})$ and $|\hat{y}| \leq r_1$ for $\hat{y} \in P$. Hence, by (3.2.1) and the assumption that Φ is increasing,

$$\begin{aligned} \Phi(a^{-1}\delta_{D_a}(\hat{y})^{-1}) &= \Phi(c_2^{-1}c_2a^{-1}\delta_{D_a}(\hat{y})^{-1}) \\ &\leq C_0(c_2^{-1})^2\Phi(a^{-1}c_2\delta_{D_a}(\hat{y})^{-1}) \leq C_0(c_2^{-1})^2\Phi(a^{-1}|\hat{y}|^{-1}). \end{aligned}$$

Thus for $\hat{y} \in P$,

$$\begin{aligned} &\Big(1 \wedge \frac{\Phi(a^{-1}|\hat{y}|^{-1})}{\Phi(a^{-1})}\Big)^{1/2} \Big(1 \wedge \frac{\Phi(a^{-1}|\hat{y}|^{-1})}{\Phi(a^{-1}\delta_{D_a}(\hat{y})^{-1})}\Big)^{1/2} \\ &\geq \Big(1 \wedge \frac{\Phi(a^{-1}(r_1)^{-1})}{\Phi(a^{-1})}\Big)^{1/2} \Big(1 \wedge c_2^2/C_0\Big)^{1/2} = c_3, \end{aligned}$$

where $c_3 = c_2/C_0^{1/2}$. By (3.2.2),

$$\Phi'(a^{-1}) = \Phi'((1+|\hat{y}|)(1+|\hat{y}|)^{-1}a^{-1}) \leq C_1(1+|\hat{y}|)\Phi'(a^{-1}(1+|\hat{y}|)^{-1}),$$

which implies

$$\frac{\Phi'(a^{-1}(1+|\hat{y}|)^{-1})}{(1+|\hat{y}|)^{d+1}} \geq C_1^{-1}\frac{\Phi'(a^{-1})}{(1+|\hat{y}|)^{d+2}} \geq C_1^{-1}\frac{\Phi'(a^{-1})}{(1+r_1)^{d+2}}.$$

Let $c_4 = C_1^{-1}/(1+r_1)^{d+2}$. Then for some $c_5 = c_5(C_0, C_1, \eta, R/d_D, d) > 0$,

$$\begin{aligned} A(x,z) &\geq c_3c_4a^{-2}\Phi'(a^{-1})\int_P \frac{\Phi'(a^{-1}|\hat{y}|^{-1})}{\Phi(a^{-1}|\hat{y}|^{-1})^2|\hat{y}|^{d+1}}\mathrm{d}\hat{y} \\ &\geq c_5\omega_d a^{-2}\Phi'(a^{-1})\int_0^{r_1}\frac{\Phi'(a^{-1}r^{-1})}{\Phi(a^{-1}r^{-1})^2r^2}\mathrm{d}r \\ &= c_5\omega_d a^{-1}\Phi'(a^{-1})\int_0^{r_1}\frac{\partial}{\partial r}\left(\frac{1}{\Phi(a^{-1}r^{-1})}\right)\mathrm{d}r \\ &= c_5\omega_d\frac{\Phi'(a^{-1})}{a\Phi(a^{-1}{r_1}^{-1})} \geq c_5\omega_{\mathrm{d}}r_1^2\frac{\Phi'(a^{-1})}{a\Phi(a^{-1})}, \end{aligned} \tag{3.2.24}$$

where the last inequality comes from (3.2.1) and $r_1 < 1$.

Therefore, from (3.2.22)–(3.2.24) we conclude that

$$K_D(x,z) \geq c_6\frac{\Phi(|x-z|^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}$$

for $c_6 = c_6(\gamma_1, C_0, C_1, C_2, C_3, C_5, \eta, R/d_D, d) > 0$. □

We now state and prove the main result of this chapter.

Theorem 3.2.6. *Let D be a bounded open set which satisfies cone condition with cone characteristic constant (R,η) and $d_D < M$ for some $M \geq 1$. Furthermore, assume that there exist a function Φ satisfying* **(P1)**-**(P4)** *and a decreasing function j such that* **(G)**, **(J1)**, **(J2)**, **(J3)** *hold. Then there exists $c = c(\gamma_1, \gamma_2, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, R/d_D, \eta, M, d) > 1$ such that* *for every $x \in D$ and $z \in \overline{D}^c$*

$$\begin{aligned}
&c^{-1}\frac{\Phi(\delta_D(z)^{-1})^{1/2}j(|x-z|)}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}\\
\le &K_D(x,z) \le c\frac{\Phi(\delta_D(z)^{-1})^{1/2}j(|x-z|)}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}.
\end{aligned} \tag{3.2.25}$$

Proof. When $z \in \overline{D}^c$ with $\delta_D(z) \le 2d_D$, By **(J2)**, (3.2.25) is equivalent to

$$K_D(x,z) \asymp \frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})}. \tag{3.2.26}$$

Indeed, when $\delta_D(z) \le 2d_D$,

$$1 \le 1+\left(\frac{\Phi(d_D^{-1})}{\Phi(\delta_D(z)^{-1})}\right)^{1/2} \le 1+\left(\frac{\Phi(d_D^{-1})}{\Phi((2d_D)^{-1})}\right)^{1/2} \le 1+2C_0^{1/2}.$$

From this and **(J2)**, we have

$$\begin{aligned}
&\frac{C_6^{-1}\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|)\\
\le&\frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})}\\
\le&\frac{C_5^{-1}C_6^{-1}(1+2C_0^{1/2})\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(R^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|),
\end{aligned}$$

which implies the equivalence between (3.2.25) and (3.2.26) for $z \in \overline{D}^c$ with $\delta_D(z) \le 2d_D$.

When $z \in \overline{D}^c$ with $\delta_D(z) > 2d_D$, we have $\delta_D(z) \le |x-z| \le 3\delta_D(z)/2$. Thus

$$(4/9C_0)\Phi(\delta_D(z)^{-1}) \le \Phi(|x-z|^{-1}) \le \Phi(\delta_D(z)^{-1}).$$

Also, we have $0 < \Phi(\delta_D(z)^{-1})^{1/2} < \Phi(d_D^{-1})^{1/2}$ from $\delta_D(z) > 2d_D > d_D$. This implies

$$
\begin{aligned}
&\frac{4\Phi(\delta_D(z)^{-1})^{1/2}}{9C_0\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|)\\
=&\frac{4\Phi(\delta_D(z)^{-1})}{9C_0\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(\Phi(\delta_D(z)^{-1})^{1/2}+\Phi(d_D^{-1})^{1/2})}j(|x-z|)\\
\leq&\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}j(|x-z|)\\
\leq&\frac{2\Phi(\delta_D(z)^{-1})}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(\Phi(\delta_D(z)^{-1})^{1/2}+\Phi(d_D^{-1})^{1/2})}j(|x-z|)\\
=&\frac{2\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}j(|x-z|).
\end{aligned}
$$

Thus (3.2.25) is equivalent to

$$
K_D(x,z)\asymp\frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(d_D^{-1})^{1/2}}j(|x-z|)
$$

when $z\in\overline{D}^c$ with $\delta_D(z)>2d_D$.

Hence by Proposition 3.2.3, Proposition 3.2.4 and (3.2.11) it suffices to show that the lower bound of (3.2.26) holds for $z\in\overline{D}^c$ with $\delta_D(z)\leq 2d_D$. For the remainder of the proof we assume $z\in\overline{D}^c$ with $\delta_D(z)\leq 2d_D$ and consider the following three cases separately.

Case 1. $R/17\leq\delta_D(z)\leq 2d_D$:

Since $|x-z|<3d_D$ and Φ is increasing, Proposition 3.2.5 implies

$$
\begin{aligned}
K_D(x,z)&\geq c_1\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi((R/17)^{-1})^{1/2}}\frac{\Phi((3d_D)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}\\
&\geq c_1c_2\frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}}\frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})},
\end{aligned}\tag{3.2.27}
$$

where $c_2=R/(C_0^{1/2}51d_D)$. Note that c_2 satisfies the inequality

$$
\Phi((R/17)^{-1})^{1/2}=\Phi((R/51d_D)^{-1}(3d_D)^{-1})^{1/2}\leq(1/c_2)\Phi((3d_D)^{-1})^{1/2}.
$$

Case 2. $|x-z|\leq 32\delta_D(z)$ and $\delta_D(z)\leq 2d_D$:

In this case, using Proposition 3.2.5 and (3.2.1) we have

$$\begin{aligned} K_D(x,z) &\geq c_1 \frac{\Phi((32\delta_D(z))^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}} \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})} \\ &\geq (c_1/32C_0^{1/2}) \frac{\Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}} \frac{\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(|x-z|^{-1})}. \end{aligned} \tag{3.2.28}$$

Case 3. $32\delta_D(z) < |x-z|$ and $\delta_D(z) < R/17$:

Define $Q := \{y \in D : |y-z| < \frac{1}{2}|x-z|\}$. For $y \in Q$,

$$|x-y| \geq |x-z| - |y-z| > |x-z| - \frac{1}{2}|x-z| > \frac{1}{2}|x-z| > |y-z|.$$

So $|x-y| > \frac{1}{2}(\delta_D(x) \vee \delta_D(y))$. This, with (3.2.1) and (3.2.4), implies that for $y \in Q$,

$$G_D(x,y) \geq c_3 \frac{1}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})}$$

for $c_3 = C_3/4C_0$. Thus by (3.2.3) and (3.2.5),

$$\begin{aligned} &K_D(x,z) \\ =&\int_D G_D(x,y)J(y-z)\mathrm{d}y \\ \geq&\int_Q \frac{\gamma_1 c_3 C_5}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})} \frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}\mathrm{d}y \\ =&\frac{\gamma_1 c_3 C_5 \Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}|x-z|^d} \\ &\times \int_Q \frac{|x-z|^d}{\Phi(\delta_D(z)^{-1})^{1/2}\Phi(\delta_D(y)^{-1})^{1/2}} \frac{\Phi'(|x-y|^{-1})}{|x-y|^{d+1}\Phi(|x-y|^{-1})} \frac{\Phi'(|y-z|^{-1})}{|y-z|^{d+1}}\mathrm{d}y \\ =:&\frac{\gamma_1 c_3 C_5 \Phi(\delta_D(z)^{-1})^{1/2}}{\Phi(\delta_D(x)^{-1})^{1/2}|x-z|^d} B(x,z). \end{aligned} \tag{3.2.29}$$

For $y \in Q$, $|x-y| \leq |x-z| + |y-z| \leq \frac{3}{2}|x-z|$. This and (P4) imply that

$$B(x,z) \geq (2/3)^{d+1} \frac{1}{\Phi(\delta_D(z)^{-1})^{1/2}|x-z|} \frac{\Phi'((3|x-z|/2)^{-1})}{\Phi((3|x-z|/2)^{-1})} \bar{B}(x,z), \quad (3.2.30)$$

where

$$\bar{B}(x,z) := \int_Q \frac{1}{|y-z|^{d+1}} \frac{\Phi'(|y-z|^{-1})}{\Phi(\delta_D(y)^{-1})^{1/2}} \mathrm{d}y.$$

Since Φ is increasing, by (3.2.2) and (3.2.30), we have

$$B(x,z) \geq C_1^{-1}(2/3)^{d+2} \frac{1}{\Phi(\delta_D(z)^{-1})^{1/2}|x-z|} \frac{\Phi'(|x-z|^{-1})}{\Phi(|x-z|^{-1})} \bar{B}(x,z). \quad (3.2.31)$$

Since D satisfies the cone condition and $\delta_D(z) < R/17 < R/4$, as in (2) in the Definition 3.1.1 there exists $z_0 \in \partial D$ and a cone $\mathcal{C}(z_0, R, \eta) \subset D$ so that $\tilde{z} = \tilde{0}$ in coordinate system CS_{z_0}. Note that $|z - z_0| \leq 2\delta_D(z)$ and $|z - z_0| = -z_d \geq 0$ in CS_{z_0}. Since $\delta_D(z) < R/17$, we have $|z-z_0| \leq 2\delta_D(z) < 2R/17 < R/8$.

We will choose $\eta' > 0$ such that

$$\begin{aligned} W &:= \{y \in B(z,(R \wedge |x-z|)/2) \setminus B(z, 2|z-z_0|) : |\tilde{y}| < \eta'(y_d - z_d)\} \\ &\subset \mathcal{C}(z_0, R, \eta/2) \cap Q. \end{aligned} \quad (3.2.32)$$

Let $\kappa = (\sqrt{3\eta^4 + 16\eta^2} - 2\eta)/(4+\eta^2)$ so that $4 = (1 + 2\kappa/\eta)^2 + \kappa^2$. Note that κ is a constant such that

$$\{(\tilde{y}, y_d) \in \partial\mathcal{C}(z_0, R, \eta/2) : |\tilde{y}| = \kappa|z-z_0|\} = \partial\mathcal{C}(z_0, R, \eta/2) \cap \partial B(z, 2|z-z_0|).$$

Let

$$1/\eta' := 1/\kappa + 2/\eta = (4+\eta^2)/(\sqrt{3\eta^4 + 16\eta^2} - 2\eta) + 2/\eta.$$

Suppose $y \in W$. First, we note that, since $|y-z| < (R \wedge |x-z|)/2 < R/2$,

$$|y - z_0| \leq |z-z_0| + |y-z| < 2\delta_D(z) + R/2 < R.$$

Now, we will prove $2|\tilde{y}| < \eta y_d$ for $y \in W$. If $|\tilde{y}| \geq \kappa|z-z_0|$, then clearly $2|\tilde{y}|/\eta \leq |\tilde{y}|/\eta' + z_d < y_d$. Suppose $|\tilde{y}| < \kappa|z-z_0|$ and $2\kappa|z-z_0|/\eta \geq y_d$.

Then using the fact that $2\kappa|z-z_0|/\eta = \kappa|z-z_0|/\eta' + z_d$ we have in CS_{z_0}

$$|y-z| = (|\tilde{y}|^2+|y_d-z_d|^2)^{1/2} < (\kappa^2|z-z_0|^2+(2\kappa|z-z_0|/\eta-z_d)^2)^{1/2} = 2|z-z_0|.$$

This is a contradiction to $y \in W$. Thus for $|\tilde{y}| < \kappa|z-z_0|$, we have $2|\tilde{y}|/\eta < 2\kappa|z-z_0|/\eta < y_d$. Hence $y \in \mathcal{C}(z_0, R, \eta/2)$, which finishes the proof of (3.2.32).

(3.2.32) implies that there exists a constant $c_4(\eta) \in (0,1]$ such that $\delta_D(y) \geq c_4|y-z_0|$ for $y \in W$. Also by definition of W, we have $|y-z| > 2|z-z_0|$ for $y \in W$. From these facts, for all $y \in W$, we have

$$\delta_D(y) \geq c_4|y-z_0| \geq c_4(|y-z|-|z-z_0|) \geq c_5|y-z|, \tag{3.2.33}$$

where $c_5 = c_4/2$. Thus by (3.2.32) and (3.2.33)

$$\begin{aligned}
&\bar{B}(x,z)\\
=&\int_Q \frac{1}{|y-z|^{d+1}}\frac{\Phi'(|y-z|^{-1})}{\Phi(\delta_D(y)^{-1})^{1/2}}\mathrm{d}y \geq \int_W \frac{1}{|y-z|^{d+1}}\frac{\Phi'(|y-z|^{-1})}{\Phi(\delta_D(y)^{-1})^{1/2}}\mathrm{d}y\\
\geq& c_6\omega_d\int_{2|z-z_0|}^{(R\wedge|x-z|)/2}\frac{1}{r^2}\frac{\Phi'(r^{-1})}{\Phi(c_5^{-1}r^{-1})^{1/2}}\mathrm{d}r\\
=&c_5c_6\omega_dC_0^{-1/2}\int_{2|z-z_0|}^{(R\wedge|x-z|)/2} -(\Phi(r^{-1})^{1/2})'\mathrm{d}r\\
=&c_5c_6\omega_dC_0^{-1/2}\Big(\Phi\big((2|z-z_0|)^{-1}\big)^{1/2}-\Phi\big(2(R\wedge|x-z|)^{-1}\big)^{1/2}\Big)
\end{aligned} \tag{3.2.34}$$

for some constant $c_6(\eta) > 0$.

For simplicity, we define

$$F(x,z) := \frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})}. \tag{3.2.35}$$

Combining Proposition 3.2.5, (3.2.29), (3.2.31), (3.2.34) and (3.2.35), for

$32\delta_D(z) < |x-z|$ and $\delta_D(z) < R/17$,

$$
\begin{aligned}
&K_D(x,z) = \frac{1}{2}K_D(x,z) + \frac{1}{2}K_D(x,z) \\
&\geq c_7 F(x,z)\frac{\Phi(|x-z|^{-1})^{1/2}}{\Phi(\delta_D(z)^{-1})^{1/2}} \\
&\quad + c_8 F(x,z)\left(\tfrac{1}{\Phi(\delta_D(z)^{-1})^{1/2}}\left(\Phi\big((2|z-z_0|)^{-1}\big)^{1/2} - 2\Phi\big((R\wedge|x-z|)^{-1}\big)^{1/2}\right)\right) \\
&\geq c_7 F(x,z)\frac{\Phi((|x-z|\wedge 3d_D)^{-1})^{1/2}}{\Phi(\delta_D(z)^{-1})^{1/2}} \\
&\quad + \ c_9 F(x,z)\left(\tfrac{1}{\Phi(\delta_D(z)^{-1})^{1/2}}\left(\Phi\big((2|z-z_0|)^{-1}\big)^{1/2} - 2\Phi\big((R\wedge|x-z|)^{-1}\big)^{1/2}\right)\right) \\
&\geq c_9\, F(x,z)\frac{\Phi\big((2|z-z_0|)^{-1}\big)^{1/2}}{\Phi(\delta_D(z)^{-1})^{1/2}} \ \geq\ c_{10}\, F(x,z). \qquad (3.2.36)
\end{aligned}
$$

In the second inequality, the constant c_9 is chosen as follows. For this, we use $|x-z| < 3d_D$. For the case $|x-z| \leq R$, take c_{11} so that $2c_{11} \leq c_7$. For $|x-z| > R$, then take c_{12} sufficiently small so that $c_7 > 2c_{12}c_{13}$, where $c_{13} = R/(3d_D C_0^{1/2})$, which satisfies $\Phi((3d_D)^{-1})^{1/2} \geq c_{13}\Phi(R^{-1})^{1/2}$. Define $c_9 = c_8 \wedge c_{11} \wedge c_{12}$. Then, the third inequality holds. For the last inequality, we use $\delta_D(z) \leq |z-z_0| \leq 2\delta_D(z)$ and so $c_{10} = c_9/4C_0^{1/2}$. Hence we get (3.2.36).

Therefore, by (3.2.27), (3.2.28) and (3.2.36), we have for $\delta_D(z) \leq 2d_D$,

$$
K_D(x,z) \geq c_{14}\frac{\Phi(\delta_D(z)^{-1})^{1/2}\Phi'(|x-z|^{-1})}{|x-z|^{d+1}\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})},
$$

where $c_{14} = c_{14}(\gamma_1, C_0, C_1, C_3, C_4, C_5, C_6, C_7, M, R/d_D, \eta, d)$. □

Corollary 3.2.7. *Suppose that $M \geq 1$ and that D is a ball with radius $r < M/2$. Furthermore, assume that there exist a function Φ satisfying* **(P1)**-**(P4)** *and a decreasing function j such that* **(G)**, **(J1)**, **(J2)**, **(J3)** *hold. Then there exists $c = c(\gamma_1, \gamma_2, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, M, d) > 1$*

such that

$$\begin{aligned}
&c^{-1}\frac{\Phi(\delta_D(z)^{-1})^{1/2}j(|x-z|)}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}\\
\le &K_D(x,z)\le c\frac{\Phi(\delta_D(z)^{-1})^{1/2}j(|x-z|)}{\Phi(\delta_D(x)^{-1})^{1/2}\Phi(|x-z|^{-1})(1+\Phi(d_D^{-1})^{1/2}\Phi(\delta_D(z)^{-1})^{-1/2})}
\end{aligned} \tag{3.2.37}$$

holds for every $x \in D$ and $z \in \overline{D}^c$. In particular, when the constants C_3, C_4 in **(G)** *are independent of $r < M/2$, then* (3.2.37) *holds for all balls with radius $r < M/2$ with the same constant c.*

Proof. For any $r < M/2$, a ball with radius r satisfies the cone condition with cone characteristic constant $(r, 1)$. Thus the ratio $R/d_D = 1/2$ and (3.2.37) holds for some $c = c(\gamma_1, \gamma_2, C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, M, d) > 1$. Except C_3 and C_4, all other constants are independent of r. Thus, if C_3, C_4 are independent of r, then the constant c is independent of the radius of the ball. □

Chapter 4

Tangential limits for harmonic functions

4.1 Main theorem

In this section we state our main theorem. To do this, we need some preparations. Throughout this chapter, we always assume that **(A-1)**–**(A-6)**. Since we consider tangential limits, we only consider $d \geq 2$.

Recall that for an open set D, $\tau_D := \inf\{t > 0 : X_t \notin D\}$. We first give the probabilistic definition of (regular) harmonic function.

Definition 4.1.1. *(1) A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic in an open set $D \subset \mathbb{R}^d$ with respect to X if for every open set B whose closure is a compact subset of D, $\mathbb{E}_x[|u(X_{\tau_B})|] < \infty$ and $u(x) = \mathbb{E}_x[u(X_{\tau_B})]$ for every $x \in B$.*
(2) A function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be regular harmonic in an open set $D \subset \mathbb{R}^d$ with respect to X if $\mathbb{E}_x[|u(X_{\tau_D})|] < \infty$ and $u(x) = \mathbb{E}_x[u(X_{\tau_D})]$ for every $x \in D$.

Clearly, a regular harmonic function in D is harmonic in D by the strong Markov property. Note that, by the Harnack inequality proved in [19], under assumptions **(A-1)**–**(A-3)** the condition $\mathbb{E}_x[|u(X_{\tau_B})|] < \infty$ for all $x \in D$ is equivalent to $\mathbb{E}_{x_0}[|u(X_{\tau_B})|] < \infty$ for some $x_0 \in D$.

The following are some function spaces related to our exterior functions:

Definition 4.1.2. *Suppose* $p \in (1, \infty]$.

(1) $\Lambda^p_\beta(\mathbb{R}^d)$ *is the space of* L^p*-Hölder continuous functions of order* β *defined on* $\mathbb{R}^d$*, i.e.,* $\bar{f} \in \Lambda^p_\beta(\mathbb{R}^d)$ *means that* $\bar{f} \in L^p(\mathbb{R}^d)$ *and there exists a constant* $c > 0$ *such that*

$$\|\bar{f}(\cdot + y) - \bar{f}(\cdot)\|_{L^p(\mathbb{R}^d)} \le c|y|^\beta \qquad \textit{for all } y \in \mathbb{R}^d. \tag{4.1.1}$$

(2) $\Lambda^p_{\beta,loc}(\overline{D}^c)$ *is the collection of functions* f *such that* f *is defined on* $\overline{D}^c$ *and for each* $\xi \in \partial D$*, there exists* $\eta > 0$ *depending on* ξ *such that* f *agrees on* $\overline{D}^c \cap B(\xi, \eta)$ *with a function in* $\Lambda^p_\beta(\mathbb{R}^d)$.

Note that functions in $\Lambda^p_{\beta,loc}(\overline{D}^c)$ may not be bounded (cf., [35, 3, 4]).
For $\gamma, a > 0$, $C^{1,1}$ open set D, and $\xi \in \partial D$, define

$$\begin{aligned} &T_{\gamma,\phi,a}(\xi) = T_{\gamma,\phi,a,D}(\xi) \\ &:= \left\{ x \in D : |x-\xi|^{\gamma+d}\phi(|x-\xi|^{-2})^{1/2} \le a \frac{\delta_D(x)^{d+2}\phi(\delta_D(x)^{-2})^{3/2}}{\phi'(\delta_D(x)^{-2})} \right\}, \end{aligned} \tag{4.1.2}$$

and $T_{\gamma,\phi}(\xi) := T_{\gamma,\phi,1}(\xi)$.

Now, we state our theorem. We use $\mathcal{H}^s$ to denote the s-dimensional Hausdorff measure on $\mathbb{R}^d$ and for a measurable subset $W \subset \mathbb{R}^d$, $|W|$ denotes the Lebesgue measure of W in $\mathbb{R}^d$.

Theorem 4.1.3. *Suppose* $p \in (1, \infty]$ *and* $\beta > 1/p$*. Let* $X = (X_t, \mathbb{P}_x)_{t \ge 0, x \in \mathbb{R}^d}$ *be a rotationally symmetric Lévy process with the characteristic exponent* $\phi(|\zeta|^2)$ *such that the assumptions* **(A-1)***–***(A-6)** *hold and* δ *in* **(A-3)** *satisfies* $1/p < \delta \le 1$*. Suppose that* D *is a* $C^{1,1}$ *open set with characteristic* (R, Λ) *and that* $f \in \Lambda^p_{\beta,loc}(\overline{D}^c)$ *satisfies* $\mathbb{E}_{x_0}[|f(X_{\tau_D})|] < \infty$ *for some* $x_0 \in D$*. Then, for* $0 < \gamma < \beta - 1/p$ *and* $a > 0$*, there exists a measurable subset* $E \subset \partial D$ *with* $\mathcal{H}^{d-1}(E) = 0$ *such that* $u_f(x) = \mathbb{E}_x[f(X_{\tau_D})]$ *has a finite limit along* $T_{\gamma,\phi,a}(\xi)$ *for every* $\xi \in \partial D \setminus E$*. Furthermore, for every* $\xi \in \partial D \setminus E$*,*

$$\lim_{T_{\gamma,\phi,a}(\xi) \ni x \to \xi} \left| u_f(x) - \lim_{r \to 0+} \frac{1}{|B(\xi, r) \setminus D|} \int_{B(\xi,r)\setminus D} f(y) \mathrm{d}y \right| = 0.$$

Note that when $\phi(\lambda) = \lambda^{\alpha/2}$ and D is the upper half-space

$$H := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\},$$

our approaching region $T_{\gamma,\phi,2/\alpha}(\xi)$ is simply $\{x \in H : |x-\xi|^{1+\gamma/(d-\alpha/2)} \le x_d\}$. Thus, our Theorem 4.1.3 covers the result stated in [30].

Since the positive constant a in (4.1.2) plays no special role in the proof, for convenience, we will only consider $T_{\gamma,\phi}(\xi) = T_{\gamma,\phi,1}(\xi)$.

Remark 4.1.4. From (2.1.3), we see that $T_{\gamma,\phi}(\xi)$ contains

$$T'_{\gamma,\phi}(\xi) := \left\{x \in D : |x-\xi|^{\gamma+d}\phi(|x-\xi|^{-2})^{1/2} \le \delta_D(x)^d \phi(\delta_D(x)^{-2})^{1/2}\right\}.$$

Moreover, $T'_{\gamma,\phi}(\xi)$ contains the Stolz open set

$$S_M(\xi) := \{x \in D : |x-\xi| \le M\delta_D(x), |x-\xi| < M^{-d/\gamma}\}$$

for $M > 1$. In fact, since $r^d\phi(r^{-2})^{1/2}$ is increasing by (2.1.2), for $x \in S_M(\xi)$, we have

$$|x-\xi|^d\phi(|x-\xi|^{-2})^{1/2} \le M^d\delta_D(x)^d\phi(M^{-2}\delta_D(x)^{-2})^{1/2} \le M^d\delta_D(x)^d\phi(\delta_D(x)^{-2})^{1/2}.$$

Thus, for $x \in S_M(\xi)$,

$$\begin{aligned} |x-\xi|^{\gamma+d}\phi(|x-\xi|^{-2})^{1/2} &\le |x-\xi|^{\gamma}M^d\delta_D(x)^d\phi(\delta_D(x)^{-2})^{1/2} \\ &< \delta_D(x)^d\phi(\delta_D(x)^{-2})^{1/2}. \end{aligned}$$

We conclude that $S_M(\xi) \subset T'_{\gamma,\phi}(\xi) \subset T_{\gamma,\phi}(\xi)$.

On the other hand, our approaching region $T_{\gamma,\phi}(\xi)$ can be strictly larger than $T'_{\gamma,\phi}(\xi)$. For example, when D is the upper half-space H and $\phi(\lambda) = \log(1+\lambda^{\alpha/2})$, where $\alpha \in (0,2]$, $d > \alpha$, we have

$$\begin{aligned} T_{\gamma,\phi}(\xi) = &\left\{x \in H : |x-\xi|^{\gamma+d}\{\log(1+|x-\xi|^{-\alpha})\}^{1/2} \right. \\ &\qquad\qquad \left. \le (2/\alpha)(1+x_d^{\alpha})x_d^d\{\log(1+x_d^{-\alpha})\}^{3/2}\right\} \\ \supset &\left\{x \in H : |x-\xi|^{\gamma+d}\{\log(1+|x-\xi|^{-\alpha})\}^{1/2} \le x_d^d\{\log(1+x_d^{-\alpha})\}^{3/2}\right\}, \end{aligned}$$

while

$$T'_{\gamma,\phi}(\xi) = \left\{x \in H : |x-\xi|^{\gamma+d}\{\log(1+|x-\xi|^{-\alpha})\}^{1/2} \le x_d^d\{\log(1+x_d^{-\alpha})\}^{1/2}\right\}.$$

4.2 Analysis on Lipschitz open set

Recall that we assume **(A-1)**–**(A-6)**. In this section, we prove some results that hold on Lipschitz open sets. We will use these results in Section 4.3.

Definition 4.2.1. *An open set D in $\mathbb{R}^d$ is said to be a Lipschitz open set if there exist a localization radius $R_{Lip} > 0$ and a constant $\Lambda_{Lip} > 0$ such that the following holds: for every $\xi \in \partial D$, there exist*

1. *a Lipschitz function $\psi = \psi_\xi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0) = 0$, $|\psi(x) - \psi(y)| \le \Lambda_{Lip}|x-y|$, and*
2. *an orthonormal coordinate system $CS_\xi : y = (\widetilde{y}, y_d)$ with its origin at ξ such that*

$$B(\xi, R_{Lip}) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R_{Lip}) \text{ in } CS_\xi : y_d > \psi(\widetilde{y})\}.$$

The pair (R_{Lip}, Λ_{Lip}) is called the characteristics of the Lipschitz open set D.

Since X is a rotationally invariant pure jump Lévy process, for every Lipschitz open set D, $\mathbb{P}_x(X_{\tau_D -} \neq X_{\tau_D}) = 1$ (see [29, 37]). Thus, for every Lipschitz open set D and every measurable function f on $\mathbb{R}^d$, which satisfies $\int_{\overline{D}^c} K_D(x_0, z)|f(z)|\mathrm{d}z < \infty$ for some $x_0 \in D$, u_f defined in Theorem 4.1.3 has the following integral representation:

$$u_f(x) = \mathbb{E}_x\left[f(X_{\tau_D}); X_{\tau_D} \in \overline{D}^c\right] = \int_{\overline{D}^c} K_D(x,z)f(z)\mathrm{d}z, \qquad x \in D. \quad (4.2.1)$$

Clearly, any regular harmonic function u in a Lipschitz open set D, whose value on D^c is f, is written as u_f.

Throughout this section, we fix the Lipschitz open set D with characteristics $(R_{\mathrm{Lip}}, \Lambda_{\mathrm{Lip}})$. Without loss of generality, we assume that $R_{\mathrm{Lip}} < 1$.

Note that D can be unbounded and disconnected. For every $\xi \in \partial D$ and $x \in B(\xi, R_{\mathrm{Lip}})$, we define the vertical distance

$$\rho_\xi(x) := x_d - \psi_\xi(\widetilde{x}),$$

where $(\widetilde{x}, x_d)$ are the coordinates of x in CS_ξ. Then,

$$\delta_D(x) \le |\rho_\xi(x)| \le (1+\Lambda_{\mathrm{Lip}})\delta_D(x), \qquad \xi \in \partial D,\ x \in B(\xi, R_{\mathrm{Lip}}). \tag{4.2.2}$$

Recall that λ_0 and δ are the constants in **(A-3)**.

Lemma 4.2.2. *For all $q \in [1, 1/(1-\delta))$, and $M \ge 1$, there exists a constant $c = c(q, \delta, \Lambda_{Lip}, M) > 0$ such that for every $\xi \in \partial D$, $s \le R_{Lip}/2$, and $r \le (2M)^{-1}(R_{Lip} \wedge \lambda_0^{-1/2})$,*

$$\int_{\{(\widetilde{y}, y_d) \text{ in } CS_\xi : |\widetilde{y}| < s, |\rho_\xi(y)| < Mr\}} \phi(\delta_D(y)^{-2})^{q/2} \mathrm{d}y \le c r s^{d-1} \phi(r^{-2})^{q/2}. \tag{4.2.3}$$

Proof. First, since ϕ is increasing, by (4.2.2), the left-hand side of (4.2.3) is less than or equal to

$$\int_{\{(\widetilde{y}, y_d) \text{ in } CS_\xi : |\widetilde{y}| < s, |\rho_\xi(y)| < Mr\}} \phi\Big((1+\Lambda_{\mathrm{Lip}})^2 |\psi_\xi(\widetilde{y}) - y_d|^{-2}\Big)^{q/2} \mathrm{d}y. \tag{4.2.4}$$

Using the assumption $q < 1/(1-\delta)$, choose $\epsilon = \epsilon(\delta, q) \in (0, \delta + 1/q - 1)$. By the change of variable $t = \rho_\xi(y)/M$, the fact that ϕ is increasing, (2.1.2), and (2.2.1),

$$
\begin{aligned}
&\int_{\{(\widetilde{y},y_d)\text{ in }CS_\xi:|\widetilde{y}|<s,|\rho_\xi(y)|<Mr\}} \phi\Big((1+\Lambda_{\mathrm{Lip}})^2|\psi_\xi(\widetilde{y})-y_d|^{-2}\Big)^{q/2}\mathrm{d}y \\
&= \phi((1+\Lambda_{\mathrm{Lip}})^2M^{-2}r^{-2})^{q/2} \\
&\qquad\times\int_{\{(\widetilde{y},y_d)\text{ in }CS_\xi:|\widetilde{y}|<s,|\rho_\xi(y)|<Mr\}} \left(\frac{\phi((1+\Lambda_{\mathrm{Lip}})^2|\psi_\xi(\widetilde{y})-y_d|^{-2})}{\phi((1+\Lambda_{\mathrm{Lip}})^2M^{-2}r^{-2})}\right)^{q/2}\mathrm{d}y \\
&\le 2(1\vee(1+\Lambda_{\mathrm{Lip}})^qM^{-q})\phi(r^{-2})^{q/2} \\
&\qquad\times\int_{\{|\widetilde{y}|<s\}}\int_0^r\left(\frac{\phi((1+\Lambda_{\mathrm{Lip}})^2M^{-2}t^{-2})}{\phi((1+\Lambda_{\mathrm{Lip}})^2M^{-2}r^{-2})}\right)^{q/2}M\mathrm{d}t\mathrm{d}\widetilde{y} \\
&\le c_1\phi(r^{-2})^{q/2}s^{d-1}\int_0^r\left(\frac{r}{t}\right)^{(1-\delta+\epsilon)q}\mathrm{d}t,
\end{aligned}
$$

which is less than or equal to $c_2\phi(r^{-2})^{q/2}s^{d-1}r$ since $(1-\delta+\epsilon)q<1$. Combining this and (4.2.4), we have proved the lemma. □

A direct consequence of Lemma 4.2.2 is the following:

Corollary 4.2.3. *For all $\delta\in(0,1]$, there exists a constant $c=c(\delta,\Lambda_{Lip})>0$ such that for every $r\le(2+2\Lambda_{Lip})^{-1}(R_{Lip}\wedge\lambda_0^{-1/2})$ and all $\xi\in\partial D$,*

$$\int_{B(\xi,r)}\phi(\delta_D(y)^{-2})^{1/2}\mathrm{d}y\le cr^d\phi(r^{-2})^{1/2}.$$

Recall that for $x\in\mathbb{R}^d$, $r>0$, and a set $W\subset\mathbb{R}^d$, $B_W(x,r)=B(x,r)\cap W$.

Lemma 4.2.4. *Suppose that $f\in\Lambda^p_{\beta,loc}(\overline{D}^c)$ and $0<\gamma<\beta-1/p$. If δ in* **(A-3)** *satisfies $\delta>1/p$, then $\mathcal{H}^{d-1}(E(\gamma))=0$, where*

$$
\begin{aligned}
E(\gamma):=\Bigg\{\xi\in\partial D:\limsup_{r\to0+}\,&r^{-2d-\gamma}\phi(r^{-2})^{-1/2}\\
&\times\int_{B_{\overline{D}^c}(\xi,r)}\int_{B_{\overline{D}^c}(\xi,r)}\phi(\delta_D(y)^{-2})^{1/2}|f(y)-f(z)|\mathrm{d}y\mathrm{d}z>0\Bigg\}.
\end{aligned}
$$

Proof. Using the cardinality, we can choose $\xi_i\in\partial D$ and $\eta_i=\eta_i(\xi_i)>0$ such that there exists $\bar{f}_i\in\Lambda^p_\beta(\mathbb{R}^d)$ satisfying $f=\bar{f}_i$ on $\overline{D}^c\cap B(\xi_i,\eta_i)$, and that,

$\partial D \subset \cup_{i\in\mathbb{N}} B(\xi_i, a_i)$, where $a_i = (R_{\mathrm{Lip}} \wedge \eta_i)/4$. Let $E_i(\gamma) = E(\gamma) \cap B(\xi_i, a_i)$ and $n_i \in \mathbb{N}$ be the largest number such that $n_i \le \log_2(a_i^{-1} \vee \sqrt{\lambda_0}) + \log_2(2 + 2\Lambda_{\mathrm{Lip}}) + 1$. For $M > 0$ and $n \ge n_i$, set

$$E_i(\gamma, M, n) = \Bigg\{ \xi \in B_{\partial D}(\xi_i, a_i) : 2^{-n(2d+\gamma)} \phi(2^{2n})^{1/2} \times \int_{B_{\overline{D}^c}(\xi, 2^{-n})} \int_{B_{\overline{D}^c}(\xi, 2^{-n})} \phi(\delta_D(y)^{-2})^{1/2} |\bar{f}_i(y) - \bar{f}_i(z)| \mathrm{d}y\mathrm{d}z > \frac{1}{M} \Bigg\}.$$

Since

$$E(\gamma) \subset \bigcup_{i=0}^{\infty} E_i(\gamma) = \bigcup_{i=0}^{\infty} \bigcup_{M=1}^{\infty} \left(\bigcap_{k=n_i}^{\infty} \bigcup_{n=k}^{\infty} E_i(\gamma, M, n) \right),$$

it suffices to show that

$$\sum_{n=n_i}^{\infty} \mathcal{H}^{d-1}(E_i(\gamma, M, n)) < \infty, \tag{4.2.5}$$

which implies $\mathcal{H}^{d-1}(\bigcap_{k=n_i}^{\infty} \bigcup_{n=k}^{\infty} E_i(\gamma, M, n)) = 0$ by Borel-Cantelli Lemma. Throughout the remainder of the proof, we fix M and i and assume that $n \ge n_i$.

Let $\psi = \psi_{\xi_i}$ and $CS = CS_{\xi_i}$ be the Lipschitz function and the orthonormal coordinate system in Definition 4.2.1. We will use this coordinate system CS below so that $\xi_i = 0$. For $\xi := (\widetilde{\xi}, \psi(\widetilde{\xi})) \in B_{\partial D}(0, a_i)$ in CS, define

$$h_n(\widetilde{\xi}) := \int_{B_{\overline{D}^c}(\xi, 2^{-n})} \int_{B_{\overline{D}^c}(\xi, 2^{-n})} \phi(\delta_D(y)^{-2})^{1/2} |\bar{f}_i(y) - \bar{f}_i(z)| \mathrm{d}y\mathrm{d}z.$$

Then, by using the area formula (see, for example, [13, Section 3.3.4]),

$$\begin{aligned}\mathcal{H}^{d-1}(E_i(\gamma,M,n)) &= \int_{B_{\partial D}(0,a_i)} \mathbf{1}_{E_i(\gamma,M,n)}(\widetilde{\xi},\psi(\widetilde{\xi}))\mathrm{d}\mathcal{H}^{d-1}(\widetilde{\xi},\psi(\widetilde{\xi}))\\ &\le M2^{n(2d+\gamma)}\phi(2^{2n})^{-1/2}\int_{B_{\partial D}(0,a_i)} h_n(\widetilde{\xi})\mathrm{d}\mathcal{H}^{d-1}(\widetilde{\xi},\psi(\widetilde{\xi}))\\ &\le M2^{n(2d+\gamma)}\phi(2^{2n})^{-1/2}\int_{|\widetilde{\xi}|<a_i} h_n(\widetilde{\xi})(1+|\nabla\psi(\widetilde{\xi})|^2)^{1/2}\mathrm{d}\widetilde{\xi}\\ &\le (1+\Lambda_{\mathrm{Lip}}^2)^{1/2}M2^{n(2d+\gamma)}\phi(2^{2n})^{-1/2}\int_{|\widetilde{\xi}|<a_i} h_n(\widetilde{\xi})\mathrm{d}\widetilde{\xi}.\end{aligned} \tag{4.2.6}$$

When $p\in(1,\infty)$, by Hölder's inequality,

$$\begin{aligned}&\int_{|\widetilde{\xi}|<a_i} h_n(\widetilde{\xi})\mathrm{d}\widetilde{\xi}\\ &\le \int_{|\widetilde{\xi}|<a_i}\int_{B(0,2^{-n})}\int_{B(0,2^{-n})}\phi(\delta_D(\xi+y)^{-2})^{1/2}|\bar{f}_i(\xi+y)-\bar{f}_i(\xi+z)|\mathrm{d}y\mathrm{d}z\mathrm{d}\widetilde{\xi}\\ &\le \left(|B(0,2^{-n})|\int_{|\widetilde{\xi}|<a_i}\int_{B(0,2^{-n})}\phi(\delta_D(\xi+y)^{-2})^{q/2}\mathrm{d}y\mathrm{d}\widetilde{\xi}\right)^{1/q}\\ &\qquad\times\left(\int_{|\widetilde{\xi}|<a_i}\int_{B(0,2^{-n})}\int_{B(0,2^{-n})}|\bar{f}_i(\xi+y)-\bar{f}_i(\xi+z)|^p\mathrm{d}y\mathrm{d}z\mathrm{d}\widetilde{\xi}\right)^{1/p}\\ &=: I\times II,\end{aligned} \tag{4.2.7}$$

where $1/q:=1-1/p$. By Fubini's theorem,

$$I^q \le c_1 2^{-nd}\int_{|\widetilde{y}|<2^{-n}}\int_{|y_d|<2^{-n}}\int_{|\widetilde{\xi}|<a_i}\phi(\delta_D(\xi+y)^{-2})^{q/2}\mathrm{d}\widetilde{\xi}\mathrm{d}y_d\mathrm{d}\widetilde{y}, \tag{4.2.8}$$

while using $|a+b|^p \le 2^{p-1}(|a|^p + |b|^p)$, the symmetry, and Fubini's theorem,

$$\begin{aligned}
II^p &= \int_{|\widetilde{\xi}|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} |\bar{f}_i(\xi+y) - \bar{f}_i(\xi+z)|^p \mathrm{d}y\mathrm{d}z\mathrm{d}\widetilde{\xi} \\
&\le 2^{p-1} \int_{|\widetilde{\xi}|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} |\bar{f}_i(\xi+y) - \bar{f}_i(\xi+y+z)|^p \mathrm{d}y\mathrm{d}z\mathrm{d}\widetilde{\xi} \\
&\quad + 2^{p-1} \int_{|\widetilde{\xi}|<a_i} \int_{B(0,2^{-n})} \int_{B(0,2^{-n})} |\bar{f}_i(\xi+y+z) - \bar{f}_i(\xi+z)|^p \mathrm{d}y\mathrm{d}z\mathrm{d}\widetilde{\xi} \\
&\le 2^p \int_{B(0,2^{-n})} \int_{|\widetilde{y}|<2^{-n}} \int_{|y_d|<2^{-n}} \int_{|\widetilde{\xi}|<a_i} |\bar{f}_i(\xi+y+z) - \bar{f}_i(\xi+y)|^p \mathrm{d}\widetilde{\xi}\mathrm{d}y_d\mathrm{d}\widetilde{y}\mathrm{d}z.
\end{aligned} \tag{4.2.9}$$

Let

$$w = (\widetilde{w}, w_d) := (\widetilde{\xi} + \widetilde{y}, \psi(\widetilde{\xi}) + y_d) = \xi + y.$$

If $|\widetilde{y}| < 2^{-n}$ and $|\psi(\widetilde{w} - \widetilde{y}) - w_d| = |y_d| < 2^{-n}$, then

$$|w_d - \psi(\widetilde{w})| \le |w_d - \psi(\widetilde{w} - \widetilde{y})| + |\psi(\widetilde{w} - \widetilde{y}) - \psi(\widetilde{w})| \le 2^{-n} + \Lambda_{\mathrm{Lip}}|\widetilde{y}| \le (1+\Lambda_{\mathrm{Lip}})2^{-n}.$$

Therefore, for $|\widetilde{y}| < 2^{-n}$,

$$\begin{aligned}
&\{w \in \mathbb{R}^d : |\widetilde{w} - \widetilde{y}| < a_i, |\psi(\widetilde{w} - \widetilde{y}) - w_d| < 2^{-n}\} \\
&\subset \{w \in \mathbb{R}^d : |\widetilde{w}| < 2a_i, |\psi(\widetilde{w}) - w_d| < (1+\Lambda_{\mathrm{Lip}})2^{-n}\} =: Q_n.
\end{aligned} \tag{4.2.10}$$

Using this and Lemma 4.2.2, the inner two integrals in (4.2.8) are bounded as

$$\begin{aligned}
\int_{|y_d|<2^{-n}} \int_{|\widetilde{\xi}|<a_i} \phi(\delta_D(\xi+y)^{-2})^{q/2} \mathrm{d}\widetilde{\xi}\mathrm{d}y_d &\le \int_{Q_n} \phi(\delta_D(w)^{-2})^{q/2} \mathrm{d}w_d\mathrm{d}\widetilde{w} \\
&\le c_2 2^{-n} \phi(2^{2n})^{q/2}.
\end{aligned} \tag{4.2.11}$$

Further, from (4.1.1), the inner two integrals in (4.2.9) are bounded as

$$\begin{aligned}&\int_{|y_d|<2^{-n}}\int_{|\widetilde{\xi}|<a_i}|\bar{f}_i(\xi+y+z)-\bar{f}_i(\xi+y)|^p\mathrm{d}\widetilde{\xi}\mathrm{d}y_d\\ &\leq\int_{Q_n}|\bar{f}_i(w+z)-\bar{f}_i(w)|^p\mathrm{d}w\leq c_3^p2^{-n\beta p}.\end{aligned}\tag{4.2.12}$$

Thus, (4.2.8), (4.2.11) imply $I\leq c_4 2^{-2nd/q}\phi(2^{2n})^{1/2}$ and (4.2.9), (4.2.12) imply $II\leq c_5 2^{-2nd/p}2^{-n(\beta-1/p)}$. From this and (4.2.7), we obtain

$$\int_{|\widetilde{\xi}|<a_i}h_n(\widetilde{\xi})\mathrm{d}\widetilde{\xi}\leq\ c_6 2^{-n(2d+\beta-1/p)}\phi(2^{2n})^{1/2}.\tag{4.2.13}$$

Now, we conclude from (4.2.6) and (4.2.13) that

$$\mathcal{H}^{d-1}(E_i(\gamma,M,n))\leq\ c_7 2^{-n(\beta-1/p-\gamma)},$$

which implies (4.2.5) since $\beta-1/p-\gamma>0$.

When $p=\infty$, simply by (4.1.1) and Corollary 4.2.3,

$$\begin{aligned}h_n(\widetilde{\xi})&\leq c_8(2^{-n+1})^\beta\int_{B_{\overline{D}^c}(\xi,2^{-n})}\int_{B_{\overline{D}^c}(\xi,2^{-n})}\phi(\delta_D(y)^{-2})^{1/2}\mathrm{d}y\mathrm{d}z\\ &\leq c_9 2^{-n(2d+\beta)}\phi(2^{2n})^{1/2}.\end{aligned}$$

Therefore, by (4.2.6), $\mathcal{H}^{d-1}(E_i(\gamma,M,n))\leq c_{10}2^{-n(\beta-\gamma)}$, which yields (4.2.5) since $\beta-\gamma>0$.

□

Lemma 4.2.5. *Suppose that $f\in\Lambda^p_{\beta,loc}(\overline{D}^c)$ and $0<\gamma<\beta-1/p$. Let*

$$F(\gamma)=\left\{\xi\in\partial D:\limsup_{r\to0+}r^{-2d-\gamma}\int_{B_{\overline{D}^c}(\xi,r)}\int_{B_{\overline{D}^c}(\xi,r)}|f(y)-f(z)|\mathrm{d}y\mathrm{d}z>0\right\},$$

then $\mathcal{H}^{d-1}(F(\gamma))=0$.

Proof. The proof of this lemma is the same as that of Lemma 4.2.4. In fact, using the same a_i, $\bar{f}_i$, n_i, and coordinate system in the proof of Lemma 4.2.4, for $n \geq n_i$, we define

$$F_i(\gamma, M, n) := \Bigg\{ \xi \in B_{\partial D}(0, a_i) : 2^{n(2d+\gamma)} \times \int_{B_{\overline{D}^c}(\xi, 2^{-n})} \int_{B_{\overline{D}^c}(\xi, 2^{-n})} |\bar{f}_i(y) - \bar{f}_i(z)| \mathrm{d}y \mathrm{d}z > \frac{1}{M} \Bigg\}.$$

When $p \in (1, \infty)$, by Hölder's inequality we have

$$\begin{aligned}
&\mathcal{H}^{d-1}(F_i(\gamma, M, n)) \\
\leq &(1 + \Lambda_{\mathrm{Lip}}^2)^{1/2} M 2^{n(2d+\gamma)} \int_{|\widetilde{\xi}|<a_i} \int_{B_{\overline{D}^c}(\xi, 2^{-n})} \int_{B_{\overline{D}^c}(\xi, 2^{-n})} |\bar{f}_i(y) - \bar{f}_i(z)| \mathrm{d}y \mathrm{d}z \mathrm{d}\widetilde{\xi} \\
\leq &(1 + \Lambda_{\mathrm{Lip}}^2)^{1/2} M 2^{n(2d+\gamma)} \left(\int_{|\widetilde{\xi}|<a_i} \int_{B(0, 2^{-n})} \int_{B(0, 2^{-n})} \mathrm{d}y \mathrm{d}z \mathrm{d}\widetilde{\xi} \right)^{1/q} \\
&\times \left(\int_{|\widetilde{\xi}|<a_i} \int_{B(0, 2^{-n})} \int_{B(0, 2^{-n})} |\bar{f}_i(\xi + y) - \bar{f}_i(\xi + z)|^p \mathrm{d}y \mathrm{d}z \mathrm{d}\widetilde{\xi} \right)^{1/p}.
\end{aligned}$$

By following the proof of Lemma 4.2.4 line by line, we see that

$$\left(\int_{|\widetilde{\xi}|<a_i} \int_{B(0, 2^{-n})} \int_{B(0, 2^{-n})} \mathrm{d}y \mathrm{d}z \mathrm{d}\widetilde{\xi} \right)^{1/q} = c_1 \left(a_i^{d-1} 2^{-2nd} \right)^{1/q},$$

and

$$\begin{aligned}
&\left(\int_{|\widetilde{\xi}|<a_i} \int_{B(0, 2^{-n})} \int_{B(0, 2^{-n})} |\bar{f}_i(\xi + y) - \bar{f}_i(\xi + z)|^p \mathrm{d}y \mathrm{d}z \mathrm{d}\widetilde{\xi} \right)^{1/p} \\
&\leq c_2 (2^{-nd} 2^{-n(d-1)} 2^{-n\beta p})^{1/p}.
\end{aligned}$$

Therefore,

$$\mathcal{H}^{d-1}(F_1(\gamma, M, n)) \leq \ c_3 2^{n(2d+\gamma)} 2^{-2nd} 2^{n/p} 2^{-n\beta} = \ c_3 2^{-n(\beta - 1/p - \gamma)}.$$

The assertion for $p \in (1, \infty)$ follows from this.

The proof for $p = \infty$ is also similar. Therefore, we skip the proof. $\square$

For a locally integrable function h on $\mathbb{R}^d$ and bounded measurable set $U \subset \mathbb{R}^d$, we define its integral mean over the region U by $⨍_U h(y)\mathrm{d}y = \frac{1}{|U|}\int_U h(y)\mathrm{d}y$.

The proof of the following lemma is taken from [30]. However, for the reader's convenience, we state the details of the proof.

Lemma 4.2.6. *Let $f \in \Lambda^p_{\beta,loc}(\overline{D}^c)$, and δ in* **(A-3)** *satisfies $\delta > 1/p$. Then,*

$$A(\xi) := \lim_{r\to 0+} ⨍_{B_{\overline{D}^c}(\xi,r)} f(y)\mathrm{d}y$$

exists and is finite for $\mathcal{H}^{d-1}$-a.e. $\xi \in \partial D$. Moreover, for $0 < \gamma < \beta - 1/p$,

$$\lim_{r\to 0+} r^{-d-\gamma}\phi(r^{-2})^{-1/2}\int_{B_{\overline{D}^c}(\xi,r)} \phi(\delta_D(y)^{-2})^{1/2}|f(y) - A(\xi)|\mathrm{d}y = 0 \qquad (4.2.14)$$

for $\mathcal{H}^{d-1}$-a.e. $\xi \in \partial D$.

Proof. For simplicity, define $A(\xi, r) = ⨍_{B_{\overline{D}^c}(\xi,r)} f(y)\mathrm{d}y$. Then, for $r \le t \le 2r$,

$$|A(\xi,t) - A(\xi,r)| \le c_1 r^{-2d}\int_{B_{\overline{D}^c}(\xi,2r)}\int_{B_{\overline{D}^c}(\xi,2r)} |f(y) - f(z)|\mathrm{d}y\mathrm{d}z.$$

Lemma 4.2.5 gives $\lim_{r\to 0+} r^{-\gamma}|A(\xi,2r) - A(\xi,r)| = 0$ for $\mathcal{H}^{d-1}$-a.e. $\xi \in \partial D$. This implies that $A_\infty(\xi) := \lim_{n\to\infty} A(\xi, 2^{-n})$ exists and

$$\lim_{k\to\infty} 2^{k\gamma}\left\{A(\xi, 2^{-k+1}) - A_\infty(\xi)\right\} = 0.$$

Therefore, for $r \le 2^{-k+1} \le 2r$,

$$\begin{aligned}
&r^{-\gamma}|A_\infty(\xi) - A(\xi,r)| \\
&\le r^{-\gamma}|A_\infty(\xi) - A(\xi,2^{-k+1})| + r^{-\gamma}|A(\xi,2^{-k+1}) - A(\xi,r)| \\
&\le 2^{k\gamma}|A_\infty(\xi) - A(\xi,2^{-k+1})| + r^{-\gamma}|A(\xi,2^{-k+1}) - A(\xi,r)| \longrightarrow 0 \quad \text{as} \quad r \to 0.
\end{aligned}$$

Thus, $A(\xi)$ exists and is finite $\mathcal{H}^{d-1}$-a.e. $\xi \in \partial D$. Further, for $\mathcal{H}^{d-1}$-a.e.

$\xi \in \partial D$,

$$\lim_{r \to 0+} r^{-\gamma} |A(\xi) - A(\xi, r)| = 0. \tag{4.2.15}$$

By Corollary 4.2.3,

$$\begin{aligned}
&r^{-d-\gamma} \phi(r^{-2})^{-1/2} \int_{B_{\overline{D}^c}(\xi,r)} \phi(\delta_D(y)^{-2})^{1/2} |f(y) - A(\xi)| \mathrm{d}y \\
&\leq r^{-d-\gamma} \phi(r^{-2})^{-1/2} \int_{B_{\overline{D}^c}(\xi,r)} \phi(\delta_D(y)^{-2})^{1/2} |f(y) - A(\xi, r)| \mathrm{d}y \\
&\qquad + c_2 r^{-\gamma} |A(\xi) - A(\xi, r)| \\
&\leq c_3 r^{-2d-\gamma} \phi(r^{-2})^{-1/2} \int_{B_{\overline{D}^c}(\xi,r)} \int_{B_{\overline{D}^c}(\xi,r)} \phi(\delta_D(y)^{-2})^{1/2} |f(y) - f(z)| \mathrm{d}z \mathrm{d}y \\
&\qquad + c_2 r^{-\gamma} |A(\xi) - A(\xi, r)|,
\end{aligned}$$

which tends to zero as $r \to 0$ for $\mathcal{H}^{d-1}$-a.e. $\xi \in \partial D$ by (4.2.15) and Lemma 4.2.4. Hence, we have proved (4.2.14). $\square$

4.3 Proof of Theorem 4.1.3

For any $C^{1,1}$ open set D with characteristic (R, Λ), it is well-known that (see, e.g., [34, Lemma 2.2]) there exists $L = L(R, \Lambda, d) > 0$ such that for every $\xi \in \partial D$ and $r \leq (R \wedge 1)$, one can obtain a $C^{1,1}$ open set $U(\xi, r)$ with characteristic $(r(R \wedge 1)/L, \Lambda L/r)$ such that

$$D \cap B(\xi, r/2) \subset U(\xi, r) \subset D \cap B(\xi, r). \tag{4.3.1}$$

First, we record a lemma, which is a consequence of the main results of [20]. Although simple, it is important in this thesis.

Lemma 4.3.1. *Let D be a $C^{1,1}$ open set with characteristic (R, Λ). Suppose that $\xi_0 \in \partial D$, $r_0 > 0$, and $u : \mathbb{R}^d \to \mathbb{R}$ is a non-negative regular harmonic function in $D \cap B(\xi_0, r_0)$ with respect to X vanishing on $D^c \cap B(\xi_0, r_0)$. Then, u vanishes continuously on $(\partial D) \cap B(\xi_0, r_0/2)$.*

Proof. Fix $\xi \in (\partial D) \cap B(\xi_0, r_0/2)$ and let $0 < r < 1 \wedge R \wedge (r_0/2)$. We will show that u vanishes continuously on $(\partial D) \cap B(\xi, r/8)$, which clearly implies the lemma.

Choose a bounded $C^{1,1}$ open set $U = U(\xi, r)$ as in (4.3.1). Let z_0 be a point in $U \setminus B(\xi, r/4)$. Then, by [20, Proposition 2.4 and Theorem 7.1], the $G_U(\cdot, z_0)$ vanishes continuously on $\partial U \supset (\partial D) \cap B(\xi, r/4)$. Moreover, $x \mapsto G_U(x, z_0)$ is a regular harmonic function in $D \cap B(\xi, r/4)$ with respect to X. Thus, by Theorem 2.2.4, for a fixed $x_0 \in B(\xi, r/8) \cap D$ and $x \in D \cap B(\xi, r/8)$,

$$u(x) \leq c u(x_0) G_U(x_0, z_0)^{-1} G_U(x, z_0) \longrightarrow 0 \quad \text{as } \ x \to (\partial D) \cap B(\xi, r/8).$$

This completes the proof. $\square$

Before we prove the main result, we observe an inequality. Recall that $g(r)$ defined in (2.1.4) is decreasing. Using this fact and the estimates in (2.2.2), we obtain that there exists $c > 0$ such that

$$\frac{\phi'(t^{-2})}{\phi(t^{-2}) t^{d+2}} \leq c \frac{\phi'(s^{-2})}{\phi(s^{-2}) s^{d+2}}, \quad s \leq t \leq 2. \tag{4.3.2}$$

Now, we prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Without loss of generality, we assume that $R < 1$. By the cardinality, we can choose $\xi_i \in \partial D$ and $\eta_i = \eta_i(\xi_i) > 0$ such that there exists $f_i \in \Lambda^p_\beta(\mathbb{R}^d)$ satisfying $f = f_i$ on $\overline{D}^c \cap B(\xi_i, \eta_i)$, and that, $\partial D \subset \cup_{i \in \mathbb{N}} B(\xi_i, \eta_i/8)$. Without loss of generality, we let $\eta_i \leq R$. Since ∂D is a countable union of $B_{\partial D}(\xi_i, \eta_i/8)$, it suffices to show that for $\mathcal{H}^{d-1}$-a.e. $\xi \in P_1 := B_{\partial D}(\xi_1, \eta_1/8)$, u_f has a limit along $T_{\gamma,\phi}(\xi)$.

Choose a $C^{1,1}$ open set $U = U(\xi_1, R)$ with characteristic $(R^2/L, \Lambda L/R)$ as (4.3.1) so that $P_1 = B_{\partial U}(\xi_1, \eta_1/8)$. Note that

$$\begin{aligned} u_f(x) &= \mathbb{E}_x[f(X_{\tau_D})] \\ &= \mathbb{E}_x[f(X_{\tau_D}); \tau_U < \tau_D] + \mathbb{E}_x[f(X_{\tau_U}); \tau_U = \tau_D] \\ &= \mathbb{E}_x[f(X_{\tau_D}); \tau_U < \tau_D] + \mathbb{E}_x[f(X_{\tau_U})] - \mathbb{E}_x[f(X_{\tau_U}); \tau_U < \tau_D]. \end{aligned}$$

By using the strong Markov property at τ_U, we see that $\mathbb{E}_x[f(X_{\tau_D}); \tau_U < \tau_D]$

and $\mathbb{E}_x[f(X_{\tau_U});\tau_U < \tau_D]$ are non-negative regular harmonic functions in U (and thus, in $B(\xi_1, R/2)\cap D$) with respect to X and vanish on $D^c\cap B(\xi_1, R/2)$. Thus, by Lemma 4.3.1, the limits of $u_f(x)$ and $\mathbb{E}_x[f(X_{\tau_U})]$ (if exist) are the same when x goes to a point $\xi \in P_1$. Therefore, it suffices to show that the limit $\lim_{T_{\gamma,\phi}(\xi)\ni x\to\xi} \mathbb{E}_x[f(X_{\tau_U})]$ exists for $\mathcal{H}^{d-1}$-a.e $\xi \in P_1$.

By Lemma 4.2.6, for $\mathcal{H}^{d-1}$-a.e. $\xi \in P_1$, $A(\xi) = \lim_{r\to 0+} \fint_{B_{\overline{U}^c}(\xi,r)} f(y)\mathrm{d}y$ exists and is finite and that

$$\lim_{r\to 0+} r^{-\gamma-d}\phi(r^{-2})^{-1/2}\int_{B_{\overline{U}^c}(\xi,r)} \phi(\delta_U(y)^{-2})^{1/2}|f(y)-A(\xi)|\mathrm{d}y = 0 \qquad (4.3.3)$$

holds. For the remainder of the proof, we fix a $\xi \in P_1$ and show that

$$\lim_{T_{\gamma,\phi}(\xi)\ni x\to\xi} |\mathbb{E}_x[f(X_{\tau_U})] - A(\xi)| = 0. \qquad (4.3.4)$$

Let $\epsilon > 0$ be given. By (4.3.3), there exists $r_0 < (1\wedge (R/4))/2$ such that for every $0 < r < 2r_0$,

$$\int_{B_{\overline{U}^c}(\xi,r)} \phi(\delta_U(y)^{-2})^{1/2}|f(y)-A(\xi)|\mathrm{d}y < \epsilon r^{\gamma+d}\phi(r^{-2})^{1/2}. \qquad (4.3.5)$$

Note that $B(\xi, 2r_0) \subset B(\xi_1, R/2)$. Let

$$u_1(x) = \mathbb{E}_x[|f(X_{\tau_U})|; X_{\tau_U} \in \mathbb{R}^d \setminus \{\overline{U}\cup B(\xi, r_0)\}],$$
$$u_2(x) = \mathbb{P}_x(X_{\tau_U} \in \mathbb{R}^d \setminus \{\overline{U}\cup B(\xi, r_0)\}).$$

Then, u_1, u_2 are non-negative regular harmonic functions in $U\cap B(\xi, r_0)$ with respect to X, and they vanish on $U^c \cap B(\xi, r_0)$.

On the other hand, since U is a bounded $C^{1,1}$ open set, we have the following Poisson kernel estimates by Theorem 3.1.3 and (2.2.4): For $x \in U$, $y \in B(\xi, r_0)\setminus \overline{U}$,

$$K_U(x,y) \le c_1 \frac{\phi(\delta_U(y)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2}\phi(|x-y|^{-2})}\frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}}. \qquad (4.3.6)$$

Thus, by (4.2.1) and (4.3.6), we have

$$
\begin{aligned}
&|\mathbb{E}_x[f(X_{\tau_U})] - A(\xi)| \\
&\leq c_1 \int_{B_{\overline{U}^c}(\xi, r_0)} \frac{\phi(\delta_U(y)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2}\phi(|x-y|^{-2})} \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}} |f(y) - A(\xi)| \mathrm{d}y \\
&\quad + u_1(x) + |A(\xi)| u_2(x). \qquad (4.3.7)
\end{aligned}
$$

Since $|x-y| \geq \delta_U(x)$ for $y \in \mathbb{R}^d \setminus \overline{U}$, by (4.3.2) and (4.3.5), for $x \in B(\xi, r_0/8) \cap T_{\gamma,\phi}(\xi)$,

$$
\begin{aligned}
&\int_{B_{\overline{U}^c}(\xi, 2|x-\xi|)} \frac{\phi(\delta_U(y)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2}} \left(\frac{\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})|x-y|^{d+2}} \right) |f(y) - A(\xi)| \mathrm{d}y \\
&\leq c_2 \frac{\phi'(\delta_U(x)^{-2})}{\delta_U(x)^{d+2}\phi(\delta_U(x)^{-2})^{3/2}} \int_{B_{\overline{U}^c}(\xi, 2|x-\xi|)} \phi(\delta_U(y)^{-2})^{1/2} |f(y) - A(\xi)| \mathrm{d}y \\
&\leq c_2 2^{d+\gamma} \frac{\phi'(\delta_U(x)^{-2})|x-\xi|^{d+\gamma}\phi(2^{-2}|x-\xi|^{-2})^{1/2}}{\delta_U(x)^{d+2}\phi(\delta_U(x)^{-2})^{3/2}} \epsilon \leq c_2 2^{d+\gamma}\epsilon. \qquad (4.3.8)
\end{aligned}
$$

When $2|x-\xi| \leq |\xi - y|$, we have $|x-y| \geq |y-\xi| - |\xi - x| \geq |\xi - y|/2$. Thus, by (4.3.2) and (2.1.3), on $\{y \in \mathbb{R}^d \setminus \overline{U} : 2|x-\xi| \leq |\xi - y| < r_0\}$,

$$
\begin{aligned}
\frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})} &\leq c_2 \frac{2^{d+2}\phi'(4|\xi-y|^{-2})}{|\xi-y|^{d+2}\phi(4|\xi-y|^{-2})} \leq \frac{c_2 2^d}{|\xi-y|^d} \\
&\leq \frac{c_2 2^{2d}}{2^d - 1}\left(\frac{1}{|\xi-y|^d} - \frac{1}{(2r_0)^d} \right) = \frac{c_2 d 2^{2d}}{2^d-1} \int_{2|x-\xi|}^{2r_0} \mathbf{1}_{\{|y-\xi|<t\}} \frac{\mathrm{d}t}{t^{d+1}}.
\end{aligned}
$$

Therefore, by the Fubini's theorem and (4.3.5), we have that for $x \in B(\xi, r_0/8)$,

$$
\begin{aligned}
&\int_{\{y \in \mathbb{R}^d \setminus \overline{U} : 2|x-\xi| \leq |\xi-y| < r_0\}} \tfrac{\phi(\delta_U(y)^{-2})^{1/2}}{\phi(\delta_U(x)^{-2})^{1/2}\phi(|x-y|^{-2})} \tfrac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}} |f(y) - A(\xi)| \mathrm{d}y \\
&\leq c_3 \phi(\delta_U(x)^{-2})^{-1/2} \int_{2|x-\xi|}^{2r_0} \left(\int_{B_{\overline{U}^c}(\xi, t)} \phi(\delta_U(y)^{-2})^{1/2} |f(y) - A(\xi)| \mathrm{d}y \right) \frac{\mathrm{d}t}{t^{d+1}} \\
&\leq c_3 \epsilon \phi(\delta_U(x)^{-2})^{-1/2} \phi(2^{-2}|x-\xi|^{-2})^{1/2} \int_0^{2r_0} t^{\gamma-1} \mathrm{d}t \\
&\leq c_3 \gamma^{-1} \epsilon (2r_0)^\gamma \leq c_3 \gamma^{-1} \epsilon, \qquad (4.3.9)
\end{aligned}
$$

where $c_3 = c_2 d 2^{2d}/(2^d - 1)$.

Applying (4.3.8) and (4.3.9) to (4.3.7), together with Lemma 4.3.1 gives

$$\limsup_{T_{\gamma,\phi}(\xi) \ni x \to \xi} |\mathbb{E}_x[f(X_{\tau_U})] - A(\xi)| \leq c_4 \epsilon,$$

where the constant $c_4 > 0$ is independent of ϵ. Since $\epsilon > 0$ is arbitrary, we have proved the claim (4.3.4). $\square$

Bibliography

[1] H. Aikawa, Harmonic functions and Green potentials having no tangential limits. *J. London Math. Soc.*, (2), **43(1)** (1991), 125–136.

[2] H. Aikawa, Fatou and Littlewood theorems for Poisson integrals with respect to non-integrable kernels. *Complex Var. Theory Appl.*, **49(7–9)** (2004), 511–528.

[3] R. F. Bass and D. You, A Fatou theorem for α-harmonic functions. *Bull. Sci. Math.* **127** (2003), 635–648.

[4] R. F. Bass and D. You, A Fatou theorem for α-harmonic functions in Lipschitz domains. *Probab. Theory Relat. Fields*, (2005), 391–408.

[5] Bogdan, K., Byczkowski, T., Kulczycki, T., Ryznar, M., Song, R. and Vondraček, Z.: *Potential analysis of stable processes and its extensions*. Lecture Notes in Mathematics, **1980**. Springer-Verlag, Berlin, 2009.

[6] K. Bogdan, T. Grzywny, M. Ryznar, Barrier, exit time and survival probability for unimodal Lévy processes, *Probab. Theory Relat. Fields*, (2014) http://dx.doi.org/10.1007/s00440-014-0568-6.

[7] K. Bogdan, T. Grzywny, M. Ryznar, Density and tails of unimodal convolution semigroups, *J. Funct. Anal.*, **266(6)** (2014), 3543–3571.

[8] K. Bogdan, T. Grzywny, M. Ryznar, Dirichlet heat kernel for unimodal Lévy processes, *Stochastic processes and their applications*, 124(2014), 3612–3650.

[9] Z.-Q. Chen, On notions of harmonicity. *Proc. Amer. Math. Soc.* **137** (2009), 3497–3510.

[10] Z.-Q. Chen and R. Song, Estimates on Green functions and Poisson kernels of symmetric stable processes, *Math. Ann.*, **312** (1998), 465-601.

[11] J. L. Doob, A relativized Fatou theorem, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 215–222.

[12] J. R. Dorronsoro, Poisson integrals of regular functions. *Trans. Amer. Math. Soc.* **297(2)** (1986), 669–685.

[13] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*. CRC press, 1992.

[14] P. Fatou, Sries trigonomtriques et sries de Taylor, *Acta Math.*, **30**(1) (1906), 335–400.

[15] N. Ikeda and S. Watanabe, On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes, *J. Math. Kyoto Univ.*, **2**(1962), 79–95.

[16] K. Kaleta, P. Sztonyk, Estimates of transition densities and their derivatives for jump Lévy processes, *preprint 2013, arXiv:1307.1302.*

[17] K.-Y. Kim and P. Kim, Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in $C^{1,\eta}$ open sets, *Stochastic processes and their applications*, **124(9)** (2014), 3055–3083.

[18] P. Kim and Y. Lee, Oscillation of harmonic functions for subordinate Brownian motion and its applications, *Stochastic Process. Appl.*, **123(2)** (2013), 422–445.

[19] P. Kim and A. Mimica, Harnack inequalities for subordinate Brownian motions, *Electronic Journal of Probability*, **17**, #37, (2012)

[20] P. Kim and A. Mimica, Green function estimates for subordinate Brownian motions : stable and beyond. *Trans. Amer. Math. Soc.*, **366(8)** (2014), 4383–4422.

[21] P. Kim, R. Song and Z. Vondraček, Boundary Harnack principle for subordinate Brownian motion. *Stoch. Proc. Appl.* **119** (2009), 1601–1631.

[22] P. Kim, R. Song and Z. Vondraček, Potential theory of subordinated Brownian motions revisited, *Stochastic analysis and applications to finance, essays in honour of Jia-an Yan*, Interdisciplinary Mathematical Sciences **13**, World Scientific, 2012, pp. 243–290.

[23] P. Kim, R. Song and Z. Vondraček, Two-sided Green function estimates for the killed subordinate Brownian motions, *Proc. London Math. Soc.* **104** (2012), 927–958.

[24] P. Kim, R. Song and Z. Vondraček, Global uniform boundary Harnack principle with explicit decay rate and its application, *Stochastic processes and their applications*, **124(1)** (2014), 235–267.

[25] P. Kim, R. Song and Z. Vondraček, Boundary Harnack principle and Martin boundary at infinity for subordinate Brownian motions, *Potential Analysis*, **41(2)** (2014), 407–441.

[26] T. Kulczycki, Properties of Green function of symmetric stable processes, *Probab. Math. Stat.* **17**(1997), 381–406.

[27] T. Kulczycki and M. Ryznar, Gradient estimates of harmonic functions and transition densities for Lévy processes, *preprint 2013, arXiv:1307.7158.*

[28] J. E. Littlewood, On a theorem of Fatou. *Proc. London Math. Soc.* **2** (1927), 172–176.

[29] P. W. Millar, First passage distributions of processes with independent increments. *Ann. Probability* **3** (1975), 215–233.

[30] Y. Mizuta, Existence of tangential limits for α-harmonic functions on half spaces. *Potential Anal.* **25** (2006), 29–36.

[31] A. Nagel, W. Rudin, and J. Shapiro, Tangential boundary behavior of functions in Dirichlet-type spaces. *Ann. Math.* **116** (1982), 331–360.

[32] A. Nagel and E. M. Stein, On certain maximal functions and approach regions. *Adv. in Math.* **54(1)** (1984), 83–106.

[33] R. Schilling, R. Song, and Z. Vondraček, *Bernstein functions: Theory and applications.* de Gruyter Studies in Mathematics 37. Berlin: Walter de Gruyter, 2010.

[34] R. Song, Estimates on the Dirichlet heat kernel of domains above the graphs of bounded $C^{1,1}$ functions. *Glas. Mat. Ser. III* **39(59)** (2004), 275–288.

[35] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, Princeton, 1970.

[36] J. Sueiro, Tangential boundary limits and exceptional sets for holomorphic functions in Dirichlet-type spaces. *Math. Ann.* **286** (1990), 661–678.

[37] P. Sztonyk, On harmonic measure for Lévy processes. *Probab. Math. Statist.* **20** (2000), 383–390.

국문초록

이 논문에서는 적분핵과 함께 특정한 적분 작용소에 대한 조화함수의 경계에서의 행동에 대해 연구한다. 먼저, 기본적인 미적분을 이용하여 기하학적 안정 과정을 포함하는 순수 점프 종속 브라운 운동에 대한 그린함수의 추정치로부터 푸아송 핵의 추정치를 계산한다. 이 같은 종속 브라운 운동의 무한소 생성자는 적분 작용소이다. 다음으로, $C^{1,1}$ 공간에서 정의된 위의 적분 작용소에 대한 조화함수가 공간 바깥에서 지수가 β인 L^p-횔더 연속인 함수이면 접선 극한이 존재한다는 사실을 얻을 수 있다.

주요어휘: 종속 브라운 운동, 그린 함수, 푸아송 핵, 적분 작용소, 조화함수, (비)접선 극한, 파투 정리
학번: 2011-30090